Finite Quantum Harmonic Oscillator*

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Abstract

The harmonic oscillator has infinities that presage those of field theory arising from its singular groups. Group regularization transforms it to a finite quantum theory based on a quantum time. The Heisenberg group, which is unstable with respect to small changes in its structure tensor, becomes the rotation group in three dimensions, which is stable. This results in pronounced violations of equipartition and of the usual uncertainty relations, and to interactions between the previously uncoupled excitation quanta of the oscillator. It freezes out the zero-point energy of extreme soft or hard oscillators, like those responsible for the infrared or ultraviolet divergencies of usual field theories, without much changing medium oscillators. It unifies and quantizes the time, energy, space, and momentum variables. In some extreme conditions this unification allows the interconversion of time and energy on a scale of 10^{51} W, the Planck power.

1 The harmonic oscillator problem

The three main evolutions of physics in the twentieth century have a suggestive family resemblance. Each introduced a small but new kind of non-commutativity, that of accelerations in special relativity, parallel transportations in general relativity, and of filtrations in quantum theory, measured in amplitude by a new fundamental constant c, G, or $\hbar$ respectively. The seminal

*Based on the Ph. D. thesis of M. Shiri-Garakani[24] suggestion of Segal [23] that stimulated the present work specified further moves toward simplicity of this gradual kind, gently modifying the commutation relations of the existing quantum theory with new quantum constants $\hbar'$, $\hbar''$ of different dimensions. These moves seem to lead, perhaps unintentionally, to a finite quantum theory and a quantum space-time, goals of some physicists since the formulation of quantum theory. They produce a theory with a simple group, having the prior theory as a limiting case and having nearly the same continuous symmetries. A special form of Segal's general concept was applied retroactively to understand the relation between special relativity and Galileo relativity [18]. More proactively, a similar regularization of the time-independent harmonic oscillator is now under study by several groups from several points of view [1, 8, 20, 21, 26, 27], and we regularize the time-dependent theory here.

When this process of gradual group modification regularizes a singular theory we call it *group regularization*. In the process one discards the prior group in favor of the regularized one. This critical step is omitted in some early studies of group "deformation," which, as the name suggests, prefer the prior group over the modified one, and retain it in the modified theory.

The linear harmonic oscillator, a constituent of all present field theories, is singular in several significant senses. By introducing Planck's quantum constant $\hbar$, Planck froze out the very stiff oscillators responsible for the infinite heat capacity of cavity radiation in Maxwell's theory, but left the zero-point energy of electromagnetism still divergent in the ultraviolet. Heisenberg later replaced the local Lagrangian and Hamiltonian of Maxwell by a reordering tailored to have zero-point energy 0 for Lorentz invariance but this is non-local and in case left the other well-known divergences of quantum electrodynamics. The quantum theory of the harmonic oscillator still carries the germ of all these divergences in that the action of its basic operators of position q, momentum p and Hamiltonian $H \sim \frac{1}{2}(p^2 + q^2)$ on almost every vector ψ in its Hilbert space diverge and are therefore undefined : $q\psi = p\psi = H\psi = \infty$. Such divergences occur in a quantum theory if and only if its Hilbert space is infinite-dimensional.

The oscillator theory is also unstable in the Segal sense detailed below. The quantum oscillator theory is not as unstable as the classical theory, which has the operators of position and momentum commute, and we have become accustomed to its divergencies, but it is still not truly operational but declares outright that its basic operations q, p can almost never be carried out.

We test group regularization on the linear harmonic oscillator, both stationary and dynamical (time-independent and time-dependent), introducing the two Segal quantum constants $\hbar', \hbar''$ besides the usual Planck constant $\hbar$, and compare the resulting finite quantum oscillators with the usual singular quantum oscillator.

Under group regularization, oscillators become rotators, the system Hilbert space becomes finite-dimensional, and the resulting operations can in principle always be carried out. Group regularization freezes out the offending zero-point oscillations of extremely hard or soft oscillators without greatly changing the zero-point energies of medium ones. The frozen oscillators also grossly violate the usual equipartition and uncertainty relations. Group regularization clearly has profound consequences for extreme energy physics: the physics of both very high and very low energies.

This toy model illustrates how a finite quantum theory of the cavity can produce a finite zero-point energy without conflicting with the many finite predictions and symmetries of the usual quantum theory. We propose that the linear harmonic field oscillators considered fundamental in present quantum physics – those of supposedly fundamental fields, not those of elastic solids, say — are actually dipole rotators in a new three-dimensional phase space, with fixed high angular-momentum quantum number l and with third angular-momentum component $m \sim l$. The unobserved oscillators responsible for the infrared and ultraviolet divergencies of present quantum theories are frozen by finite quantum effects described here and contribute negligibly to the zero-point energy.

2 Group flexing and flattening.

We review some standard concepts leading to the Lie algebraic concepts of semi-simple group and radical. We write the non-associative Lie product and the commutator product $ab - ba$ also as $a \times b = [a, b] = \triangle a \cdot b$. Here $\triangle a$ is a linear operator on the vector space $\mathcal{L}$ for each $a \in \mathcal{L}$. The Killing form on $\mathcal{L}$ is $\|a\| = \mathrm{T}r\,(\triangle a)^2$. If $\mathcal{L}$ is a Lie algebra, one defines the derived algebra $\partial\mathcal{L} = \mathcal{L}\times\text{'}\,\mathcal{L}$ as the Lie subalgebra with the space $\mathcal{L}\times\text{'}\,\mathcal{L}$. A Lie algebra $\mathcal{L}$ is *solvable* if $\exists n \in \mathbb{N} : \partial^n\mathcal{L} = \{0\}$. The *radical* of a Lie algebra is its maximal solvable ideal.. A *semi-simple* Lie algebra is one with vanishing radical. Equivalently, a semi-simple Lie algebra has non-singular Killing form. An algebra that is not semi-simple we call *compound*. These terms apply also to a Lie group whenever they apply to its Lie algebra.

Lie products $\times$ on a given vector space $\mathcal{V}$, also called structure tensors, form a quadratic sub-manifold $\mathcal{J}(\mathcal{V})$ defined by the Jacobi law, which is quadratic in $\times$, in the tensor space $\mathcal{V}\otimes[\mathcal{V}^\dagger\otimes\mathcal{V}^\dagger]$; here the † dualizes, the brackets skew-symmetrize, and $\mathcal{J}(\mathcal{V})$ imports its topology from the surrounding tensor space $\mathcal{V}\otimes[\mathcal{V}^\dagger\otimes\mathcal{V}^\dagger]$.

Now the key definition: A *regular* (stable, robust, generic, . . .) Lie algebra is one whose structure tensor or Lie product $\times$ is isomorphic to all those in some neighborhood $N(\times) \subset \mathcal{J}(\mathcal{V})$.

A Lie algebra is regular if and only if its Killing form is regular [23].

Thus the Lorentz group of space-time transformations is stable. For example, small corrections to light-speed c do not change the structure of this group.

By group *flexing* we mean a homotopy of the structure tensor of a compound group that makes it semi-simple. Group *flattening* is the inverse process. The well-known contraction process of Inönü and Wigner [18] is a special case of flattening accomplished by a one-parameter group of dilations of a coordinate system of the Lie algebra in a fixed direction. The inverse to group contraction is *group expansion* [17] and is a special case of flexing.

Several regularization processes have been used to remove unwanted infinities from physics. Unphysical regularizations seek to cope with the divergencies of a theory without changing the finite results. They are regarded as giving the theory meaning rather than changing the theory. These include Pauli-Villars regularization [22], lattice regularization (lattice gauge theory) [28] and dimensional regularization [6, 7, 26]. They contain regularization parameters that go to singular limits, such as lattice spacings that go to 0. Physical regularizations, on the contrary, are new physical theories in their own right. They change the finite predictions as well as making the infinite ones finite. Their regularization parameters do not go to a singular limit but must be determined by experiment. The most famous example is Planck's, which ultimately led to quantum theory. Heisenberg's regularization left the Killing form of the (q, p) Lie algebra 0. To eliminate this 0 takes two new quantum constants.

The group regularization we carry out here is proposed as a toy model of a physical regularization.

Compound groups are *unstable* with respect to a small change in their structures [23]. Flexing stabilizes them. Conversely, flattening destabilizes. Approximating a circle by a tangent line or a sphere by a tangent plane are well-known flattenings. The circle and sphere are finite and their flattened form is infinite. Finite dimensional representations of the group of the sphere — such as spherical harmonic polynomials — form a complete set on the sphere, and all the operators of an irreducible representation have finite bounded spectra. On the other hand the tangent plane is not compact and requires infinite-dimensional representations of the translation group for a complete set, and its group generators have unbounded spectra.

3 Simplifying the Heisenberg Algebra

The Heisenberg Lie algebra $\mathcal{H}1$ is defined by the commutation relations

$$p \times q = -i\hbar 1, \quad 1 \times q = 0, \quad q \times 1 = 0. \tag{1}$$

among its three generators $q, p, 1$. It is compound and solvable, its own radical. Segal proposed to simplify $\mathcal{H}1$ by replacing 1 with a variable o similar to q and p, with two more quantum constants, which we designate here by $\hbar' \equiv \hbar^{[1]}$ and $\hbar'' \equiv \hbar^{[2]}$, and flexed commutation relations, notation changed,

$$\widehat{q}\times\widehat{p} = \hbar i\widehat{r}, \quad \widehat{r}\times\widehat{q} = \hbar' i\widehat{p}, \quad \widehat{r}\times\widehat{p} = -\hbar'' i\widehat{q}, \tag{2}$$

[8, 27, 23] These define the group SO(2, 1). The irreducible unitary representations of this non-compact group are infinite-dimensional. Ultimately we will need an indefinite metric for relativistic reasons, but not for the time-independent harmonic oscillator. To avoid these infinities, we change one sign. Let $q \equiv q^{[0]}$, $p \equiv q' \equiv q^{[1]}$, $o \equiv q'' \equiv q^{[2]}$. We assume an invariant Euclidean metric g_{ij} and use the regularization [4, 16]

$$\widehat{q}^{[i]} \times \widehat{q}^{[j]} = -i \sum_k \epsilon^{ij}{}_k \hbar^{[k]} \widehat{q}^{[k]}. \tag{3}$$

We take $\hbar, \hbar', \hbar'' > 0$ so that the orthogonal group is SO(3). Quantities with a circumflex like $\widehat{q}$ are the flexed quantum operators.

Group regularization generally creates the same kind of factor-ordering problems that quantization did.

Except for scale factors the simplified commutation relations are those of an SO(3) quantum angular-momentum operator-valued vector $\mathbf{L} = i\mathbf{L}\times\mathbf{L}$ for a dipole rotator in three-dimensional space. We assume an irreducible representation with

$$\mathbf{L}^2 = l(l+1) \tag{4}$$

where l can have any non-negative half-integer eigenvalue. (In the present work it suffices to consider only integer values of l.) Then L_1, L_2, L_3 are represented by $(2l+1)\times(2l+1)$ matrices obeying

$$L^i \times L^j = -i\epsilon^{ijk} L_i. \tag{5}$$

We fix the scale factors by setting

$$q^{[k]} = Q^{[k]} L^{[k]} \quad \text{(No sum over } k.) \tag{6}$$

In the singular limit $l \to \infty$ and the oscillator is nearly polarized along the L_3 axis, with $L_3 \approx l$. By (5)

$$Q = \sqrt{\hbar\hbar'}, \quad Q' = \sqrt{\hbar\hbar''}, \quad Q" = \sqrt{\hbar'\hbar''} \;=\; 1/l. \tag{7}$$

The commutation relations (5) and the angular momentum quantum number l determine a simple (associative) enveloping algebra Alg($\mathbf{L}, l$) The spectral spacing of L_3 is 1, so the finite quantum constants $Q, Q', Q"$ serve as quanta of the position, momentum and action variables. Since q, p are supposed to have continuous spectra in quantum theory, the constants $Q, , Q'$ must be very small on the ordinary quantum scale. It follows that $Q" = QQ'/\hbar$ is also very small on that scale and $l \gg 1$.

For $l \gg \sqrt{l} \gg 1$, variations $\delta(o^2) \leq O(l^{-1/2}) \ll 1$ about $o^2 = 1$ can be negligible at the same time as the spectral intervals $\delta p \leq Q'\sqrt{l}$ and $\delta q \leq Q\sqrt{l}$ for quasicontinuous $p, q \approx 0$. This simulates the usual oscillator.

4 Stationary harmonic oscillator

In this section we regularize the time-independent linear harmonic oscillator (§4), with quantum coordinate q and momentum p, and in a later section the time-dependent one (§5), with additional differential operators t and ∂_t.

A familiar ordering question arises at once. Since group regularization introduces non-commutativity, the factor-ordering of some products is immaterial in the unregularized theory but is significant in the regularized theory. Re-ordering the singular theory will introduce a correction of order $h'h''$ that may be experimentally significant. This makes the group regularization process as notation-dependent and ambiguous as the canonical quantization process.

Here there is no dynamical law for us to flex, just the canonical commutation relations, the Hamiltonian operator H, and the ground state. We regularize the Heisenberg group $\mathbf{H}(1)$, generated by the Lie algebra of $q, p, 1$, and the unitary group $U(\infty)$ of the associative algebra generated by q, p and the canonical commutation relation. We ignore time as much as possible.

We specialize to the finite harmonic oscillator by fixing the Hamiltonian

$$H = \frac{Q'^2}{2m} L_2^2 + \frac{kQ^2}{2} L_1^2 =: \frac{K}{2} \left({L_2}^2 + \kappa^2 {L_1}^2 \right) \tag{8}$$

where

$$K := \frac{(Q')^2}{m}, \quad \kappa^2 = \frac{\hbar' mk}{\hbar''}. \tag{9}$$

We may think of κ as a dimensionless oscillator freqency. For fixed $\hbar^{[k]}$, all finite oscillators are divided into three kinds with ill defined boundaries: *medium*, where kinetic and potential terms in H are of comparable size ($\kappa \sim 1$); *soft*, with $\kappa \to 0$); and *hard*, with $\kappa \to \infty$. For a spin-zero scalar field in the singular quantum field theory, the oscillators that give rise to infrared divergencies become soft oscillators in the finite quantum theory, and those that feed ultraviolet divergencies become hard oscillators.

4.1 Medium oscillators

The case $\kappa = 1$ is symmetric under rotations about the z axis, and so is especially simple [26]. Since

$$(L_1)^2 + (L_2)^2 + (L_3)^2 = (L^2), \tag{10}$$

it follows that

$$H = \frac{K}{2}\left(l(l+1) - (L_3)^2\right) \tag{11}$$

The oscillator quantum number n that labels the energy level is now

$$n = l + m. \tag{12}$$

The expanded energy spectrum is

$$E_n = \frac{K}{2}\left(l(l+1) - (n-l)^2\right) = lK\left(n + \frac{1}{2} - \frac{n^2}{2l}\right) \tag{13}$$

For $n \ll \sqrt{l} \ll l$ this reproduces the usual uniformly-spaced oscillator energy spectrum as closely as desired, but with multiplicity 2 for each level instead of 1.

The ground-state energy for this oscillator is

$$E_0 = \frac{1}{2}Kl = \frac{1}{2}1/2\hbar\omega, \tag{14}$$

exactly the usual oscillator ground energy, since $Kl = \hbar\omega$.

The main new feature is that this finite oscillator has an upper energy limit

$$E_{max} = \frac{1}{2}Kl(l+1) \tag{15}$$

as required by a finite quantum theory.

In the general case of $\kappa \sim 1$ we obtain an upper bound for the ground energy by a variational approximation with the trial function $|L_3 = \pm l\rangle$. This reproduces our previous result (14), now as an upper bound for the ground energy of a medium FLHO:

$$E_0 \leq \frac{1}{2}Kl. \tag{16}$$

Medium oscillators have many states with m-value close to the extremum values $m = \pm l$. The usual Heisenberg uncertainty principle

$$(\Delta p)^2(\Delta q)^2 \geqslant \frac{1}{4}\langle ip \times q\rangle^2 = \frac{\hbar^2}{4}. \tag{17}$$

becomes

$$(\Delta L_1)^2(\Delta L_2)^2 \geqslant \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L_3\approx\pm l\rangle} \tag{18}$$

for a low-lying energy level of a medium oscillator. By (6) and (7),

$$(\Delta p)^2(\Delta q)^2 \geqslant \frac{\hbar^2}{4} \tag{19}$$

for large l. So medium oscillators in low-lying energy levels have uncertainties obeying the Heisenberg uncertainty principle.

4.2 Soft oscillators

When $\kappa \ll 1$ we can estimate the spectrum of the finite Hamiltonian (8) by perturbation theory. The unperturbed Hamiltonian is now the kinetic energy

$$H_0 = \frac{K}{2}L_1^2 \tag{20}$$

and the unperturbed eigenvectors are $|L_1 = m\rangle$ so the unperturbed energy levels are

$$E_m(0) = \frac{K}{2}m^2. \tag{21}$$

The first-order shifts are

$$\delta E_m = \frac{K}{2}\langle L_1 = m|L_2^2|L_1 = m\rangle. \tag{22}$$

Due to the axial symmetry of $|L_1 = m\rangle$,

$$\langle L_1 = m|L_2^2|L_1 = m\rangle = \langle L_1 = m|L_3^2|L_1 = m\rangle. \tag{23}$$

Therefore the energy shift is, to lowest order in κ^2,

$$\begin{aligned}\frac{K}{2}\langle L_1 = m|\kappa^2 L_2^2|L_1 = m\rangle &= \frac{K}{4}\kappa^2\langle m|L_1^2 + L_2^2|m\rangle \\ &= \frac{K}{4}\kappa^2\langle m|L^2 - L_3^2|m\rangle \\ &= \frac{K}{4}\kappa^2 l(l+1) - m^2 \end{aligned} \tag{24}$$

The new energy spectrum is then

$$\begin{aligned} E_m &\approx \frac{K}{2}m^2 + \Delta E_m \\ &= \frac{K}{2}m^2 + \frac{1}{4}K\ \kappa^2 \left[l(l+1) - m^2\right] \end{aligned} \tag{25}$$

The estimated upper bound for the energy is

$$E_{max} \approx \frac{1}{2}Kl^2(1 + \frac{\kappa^2}{2l}) \tag{26}$$

For $\kappa \to 0$ this reproduces the upper bound for the unperturbed Hamiltonian L_3^2, as it should. The zero-point energy E_0 of first-order perturbation theory is

$$E_0 \approx \frac{1}{4}\kappa^2 Kl(l+1) \tag{27}$$

For $\kappa \to 0$ this is infinitesimal compared to the usual singular quantum oscillator.

A soft oscillator shows no resemblance to the usual quantum oscillator. Its energy levels do not have uniform spacing. Its kinetic energy dwarfs its potential energy, so equipartition is grossly violated. The low energy states are near $|L_1 = 0\rangle$ instead of $|L_3 = \pm l\rangle$. Its p degree of freedom is frozen out. It is "too soft to oscillate:" There is not enough energy in the q degree of freedom, even at its maximum excitation, to produce one quantum of p. The uncertainty relation reads

$$(\Delta L_1)^2(\Delta L_2)^2 \geqslant \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L_1\approx 0\rangle} \approx 0 \tag{28}$$

Therefore

$$\Delta p \Delta q \ll \frac{\hbar}{2}, \tag{29}$$

which violates the Heisenberg uncertainty principle grossly.

4.3 Hard oscillators

Hard oscillators reverse the story but violate the same basic principles of the singular quantum theory as soft oscillators. A hard oscillator has much greater potential than kinetic energy. Its low energy states are now near $|L_2 = 0\rangle$ instead of $|L_3 = \pm l\rangle$ (the medium case) or $|L_1 = 0\rangle$ (the soft case). There is not enough energy in the p degree of freedom, even at maximum excitation, to excite one quantum of q. Its q degree of freedom is frozen out. It is "too hard to oscillate."

A hard oscillator too can be treated by perturbation theory. Now the kinetic energy is the perturbation. We may carry all the of the main results in the previous section for soft oscillators to the hard ones simply by replacing κ with $1/\kappa$ and K with $K\kappa^2$. A hard oscillator shows no resemblance to the usual quantum oscillator. Its zero-point energy E_0 is now

$$E_0 \approx \frac{K}{4} l(l+1) \tag{30}$$

For $\kappa \to \infty$ this is infinitesimal compared to the usual quantum oscillator zero-point energy. The energy levels of a hard oscillator are not uniformly spaced. Its uncertainty relation reads

$$(\Delta L_1)^2 (\Delta L_2)^2 \geqslant \frac{\hbar^2}{4} \langle L_3 \rangle^2_{|L_2 \approx 0\rangle} \approx 0 \tag{31}$$

Therefore

$$\Delta p \Delta q \ll \frac{\hbar}{2}, \tag{32}$$

which seriously violates the Heisenberg uncertainty principle again.

4.4 Unitary Representations

Variables p and q do not have finite-dimensional unitary representations in classical and quantum physics. They are continuous variables and generate unbounded translations of each other. But since in the finite quantum theory, all operators become finite and quantized, we expect all translations to become rotations with simple finite-dimensional unitary representations.

The canonical group of a classical oscillator becomes the unitary group of an infinite-dimensional Hilbert space for a quantum oscillator, and the unitary group of a $2l+1$ dimensional Hilbert space for the finite oscillator.

The Lie algebra generated by momentum and position as infinitesimal symmetry generators is $\mathbf{H}(1)$ for the classical and quantum oscillator and the $\mathbf{SO}(3)$ angular momentum algebra for the finite oscillator. The corresponding Lie groups are the Heisenberg group $\mathcal{H}(1)$ and the orthogonal group $\mathbf{SO}(3)$.

The commutation relations $\mathbf{L} \times \mathbf{L} = -i\mathbf{L}$ and the angular momentum quantum number l determine a simple matrix algebra $\mathrm{Alg}(\mathbf{L}, l)$; here l can be any non-negative half-integer. The spectral spacing of the operators L_k is 1, so the constants $\hbar_k$ serve as quanta of the $\widehat{q}_k$ respectively. Since q, p have continuous spectra in the flattened quantum theory, the quanta $\hbar_1, \hbar_2$ of the flexed operators $\widehat{q}, \widehat{p}$ must be small on the ordinary quantum scale $\hbar \sim 1$. It follows that $\hbar_3$ is also small on that scale.

For $\sqrt{l} \gg 1$, variations $\delta(\widehat{r}^2) \leq O(l^{-1/2}) \ll 1$ from $\widehat{r}^2 = 1$ can be negligible at the same time as the spectral intervals $\delta p \leq \hbar_2 \sqrt{l}$ and $\delta q \leq \hbar_1 \sqrt{l}$ for quasicontinuous $p \ll \hbar_2, q \ll \hbar_1$.

4.5 Quantum internal space

It is natural to ask what space the oscillator moves in. In c physics we usually describe a space as a set of space-points with some structure. But in quantum theory we describe a space by its algebra of coordinates, contragredient to the usual notion. The coordinate space of the q oscillator is defined in the quantum sense by the complex commutative algebra $\mathrm{Alg}[q, \mathbb{C}]$ generated by the coordinate operator q. The phase space of the same q oscillator is defined by the complex non-commutative Heisenberg algebra $\mathcal{H}(1) = \mathrm{Alg}[q, p, \mathbb{C}]$. Both algebras are infinite dimensional. We say nevertheless that the oscillator coordinate space is 1-dimensional, because its algebra is generated by the one operator q, with spectrum $\mathbb{R}$. The q oscillator phase-space can still be called 2-dimensional because its algebra has two independent generators q, p, each with spectrum $\mathbb{R}$.

In the finite quantum oscillator theory, where $\hbar i \in \mathbb{C}$ has been replaced by $\hbar i \widehat{r}$, we naturally define the corresponding quantum phase-space by the complex algebra $\mathrm{Alg}[\widehat{q}, \widehat{p}, \widehat{r}] = \mathrm{lg}\ \mathrm{SO}(3)$. This slight change in the commutation relations changes the spaces drastically in the large. The three variables L_1, L_2, L_3 are on the same footing, and are related to the constant total quantum number l by $(L_1)^2 + (L_2)^2 + (L_3)^2 = l(l+1)$.

The Hamiltonian (8) is that of a rigid dipole rotator with one infinite principle moment of inertia $I_3 = \infty$ and with a sharp total angular momentum quantum number $l < \infty$. It has a finite number of states $N = 2l + 1 \sim 1/(\hbar' \hbar'')$. Manfredi and Salasnich discuss the statistics of the energy spectrum of the triaxial rotator and point out that part of the rotator spectrum can approximate the spectrum of a linear harmonic oscillator [21].

Obviously the operator $\hbar\omega(L_1 2 + l))$ has exactly the energy spectrum of the singular theory, cut off at the Nth level. There is presumably a modification of $\widehat{H}'$ of $\widehat{H}$ that has exactly this equally spaced spectrum and differs from our quadratic $\widehat{H}$ by corrections that vanish in the singular limit. This equally spaced energy spectrum eliminates the interactions between the quanta of excitation when their number is less than N, while the impossibility of higher occupation than N can be regarded as an effective infinite repulsive $\mathsf{N}+1$-body potential.

4.6 Comparison of regular and singular quantum oscillators

Let us compare the classical, quantum, and finite linear harmonic oscillators [24].

Every finite quantum oscillator is isomorphic to a dipole rotator with Hamiltonian of the special form

$$H = \frac{1}{2}K_x(L_x)^2 + K_y\frac{1}{2}(L_y)^2, \quad K_x = \frac{\mathsf{P}^2}{\mu}, \quad K_y = \frac{\mathsf{Q}^2}{\lambda}. \tag{33}$$

The singular classical and quantum oscillators have continuous coordinates q and momenta p. The finite-oscillator position and momentum variables $\widehat{q}, \widehat{p}$ are quantized with finite, uniformly spaced, spectra, with spacing Q, P respectively, and maximum values $l\mathsf{Q}, l\mathsf{P}$. Systems presently recognized as oscillators have many states, $\mathsf{N} = 1/\mathsf{J} \gg 1$ in number. Oscillators with $\mathsf{N} \sim 1$ are not recognized today as oscillators but are recognized as rotators at once.

In the classical theory all oscillators are isomorphic up to scale. All singular quantum linear harmonic oscillators are likewise isomorphic up to scale. The constants $\hbar, \hbar''$ finally break this scale invariance. The finite quantum linear harmonic oscillators fall into three broad classes, which we term *soft*, *medium*, and *hard*, according to the dimensionless ratio $K_y/K_x = \kappa^2$ of maximum possible potential energy to maximum possible kinetic energy.

Medium oscillators ($\kappa \sim 1$) have $\sim \sqrt{N}$ low-lying states with nearly the same zero-point energy and level spacing as the singular quantum oscillator, like rotators nearly polarized along the z axis with $L_z \sim \pm l$. They resemble the singular quantum oscillator in that they obey the Heisenberg uncertainty principle and the equipartition principle when they are in their low-lying energy levels.

The soft and hard finite oscillators do not resemble the singular quantum oscillator at all. Their low-lying energy states correspond to rotators with $\kappa \sim 0$ or $\kappa \sim \infty$. Their 0-point energy is infinitesimal compared to the singular quantum oscillator. They grossly violate both the uncertainty principle and equipartition in all their states.

Soft oscillators have frozen momentum $p \sim 0$, their maximum potential energy being too small for even one quantum of momentum.

Hard oscillators (kinetic $\ll$ potential) have frozen position $q \sim 0$, their maximum kinetic energy being too small for even one quantum of position. These quantum freezings of degrees of freedom resemble but extend the original ones by which Planck obtained a finite thermal distribution of cavity radiation. Even the 0-point energy of a similarly regularized field theory will be finite, and can therefore be physical.

Straightforward regularization of the singular harmonic oscillator results in the energy-constraint operator

$$\widehat{E} - \widehat{H} = AL_{14} - BL_{23}^2 - CL_{24}^2 \tag{34}$$

with real positive constants A, B, C. For medium oscillators this approaches the singular theory as $\hbar_{rp}, \hbar_{rq} \to 0$. For soft oscillators the mass dominates the spring, $C \ll B$ and may be treated as a perturbation. The kinetic energy perturbation CL_{24}^2 happens to commute with the unperturbed energies $AL_{14} - BL_{23}^2$.

5 Dynamical harmonic oscillator

Having regularized the stationary or time-independent quantum oscillator, a theory with no dynamics, we now regularize the dynamical or time-dependent one, staying as close to the singular theory as regularity permits. Above all we maintain and extend the correspondence principle. The variables and equations of the flexed theory converge (non-uniformly) to those of the singular theory in the quantum limit.

This is a critical test of the strategy of group regularization. We have not succeeded in making a reasonable dynamical theory of a q/q system before now.

As a preliminary to regularization by simplification we reformulate the singular time-dependent theory algebraically.

5.1 Timed operators

In ordinary usage, each element of the system (associative) algebra ($\mathcal{S}$) represents an entire equivalence class of physical acts of determination possibly at different times equivalent by virtue of the dynamics. One observable can be measured at any time by suitable timed actions if a definite Hamiltonian is operative. For brevity we speak of operator-valued functions of time as *timed* operators, and speak of what they represent as timed variables and timed actions on the system. The timed operators form an algebra $\mathcal{A}(\mathcal{S}, \mathcal{T})$ that we construct as follows. Let $\mathbf{T}$ designate the real field $\mathbf{R}$ as affine time axis. The commutative algebra of power series in the time variable $t \in \mathbf{T}$ we designate by $\mathcal{A}(\mathcal{T})$. Then the algebra of timed operators of the Heisenberg quantum theory is $\mathcal{A}(\mathcal{S}, \mathcal{T}) = \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(\mathcal{T})$. Since both factor algebras are unital, they are sub-algebras of their product.

The time coordinate itself is not an observable in the operator algebra $\mathcal{A}(\mathcal{S})$ but a timed operator in $\mathcal{A}(\mathcal{S}, \mathcal{T})$. The operator d/dt of the Heisenberg dynamics is not even a timed operator but an operator on timed operators: $d/dt : \mathcal{A}(\mathcal{S}, \mathcal{T}) \to \mathcal{A}(\mathcal{S}, \mathcal{T})$. In the Heisenberg equation (35), both the variable $q(t)$ and the Hamiltonian operator H are timed operators in $\mathcal{A}(\mathcal{S}, \mathcal{T})$.

The Heisenberg dynamical equations

$$i\hbar \frac{dq(t)}{dt} - H \times q(t) = [E - H] \times q(t) = 0. \tag{35}$$

concern timed operators $q(t)$, not mere operators or observables q. We designate by $\mathcal{A}(\mathcal{S},\mathcal{T};H) \subset \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(\mathcal{T})$ the *dynamical algebra*, consisting of timed operators q obeying the Heisenberg equation with the specified Hamiltonian H, and call its timed operators *dynamical* variables or operators. The commutation relations $q_1(t_1) \times q_2(t_2) = q_3(t_1,t_2)$ of dynamical variables at different times depend on the Hamiltonian.

To stabilize the dynamical theory we flex and stabilize the timed algebra $\mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(\mathcal{T})$ and the dynamical equations in turn.

To designate for the Lie algebras generated for example by the same basic variables q,p as $\mathcal{A}(\mathcal{S})$ with the commutator as Lie product, we replace $\mathcal{A}(\mathcal{S})$ by $\mathcal{L}(\mathcal{S})$. The much bigger Lie algebra that is the commutator algebra of an associative algebra $\mathcal{A}(\mathcal{S})$ we designate by $\mathcal{L}\mathcal{A}(\mathcal{S})$.

5.2 Flexing the timed algebra

We flex the time space $\mathcal{T}$ following the precedent of the system space $\mathcal{S}$. To regularize the Killing form of $\mathcal{A}(\mathcal{T})$ most economically we first adjoin the operator d/dt, forming an interim Heisenberg algebra that we regard as defining a quantum time space $\mathcal{S}(t,E)$ with non-commutative time-energy operators t and $E := i\hbar d/dt$ generating a Lie algebra $\mathcal{L}(t,-E) \sim \mathrm{lg}\mathcal{H}(1)$. Now the partial time derivative of any timed operator $V \in \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(t,E)$ is $\partial_t V = d/dt \times V$.

A notational reminder: The timed energy operator $E := i\hbar\partial_t \in \mathcal{A}(t,E) \subset \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(t,E)$ is mathematically distinct from the Hamiltonian $H \in \mathcal{A}(\mathcal{S}) \subset \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(t,E)$.

When we combine two independent physical systems $\mathcal{S}_1, \mathcal{S}_2$ into one composite system $\mathcal{S} = \S_1\S_2$, the Lie algebra $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ of the composite is a commutative direct sum of the two partial Lie algebras $\mathcal{L}_1, \mathcal{L}_2$. That is, the composite vector space is a direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, and $\mathcal{S}_1$ operators act on the subspace $\mathcal{V}_1$ and commute with $\mathcal{S}_2$ operators acting on the subspace $\mathcal{V}_2$. The associative operators algebras combine by the tensor product: $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$.

In the example of the linear harmonic oscillator, we form the commutative direct sum of the Heisenberg kinematical Lie algebra $\mathcal{L}(q,p)$ of a time-independent oscillator theory with a Heisenberg time-energy Lie algebra $\mathcal{L}(t,-E)$ to form the timed Lie algebra $\mathcal{L}(\mathcal{S}) \oplus \mathcal{L}(t,-E)$ of the time-dependent theory of a singular time-dependent oscillator. The kinematical timed algebras are then tensor-multiplied:

$$\mathcal{A}(\mathcal{S},t,-E = \mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(t,-E). \tag{36}$$

The Heisenberg dynamical equation now takes the form

$$\triangle(E-H) \cdot V = 0. \tag{37}$$

We designate the dynamical sub-algebra of $\mathcal{A}(\mathcal{S},\mathcal{T})$ by $\mathcal{A}(\mathcal{S},\mathcal{T} : E-H)$. It is the centralizer of $E-H$ in the timed algebra $\mathcal{A}(\mathcal{S})\otimes\mathcal{A}(t,E)$. It depends on the Hamiltonian.

In the time-independent theory we flexed the canonical pair (q,p) to a canonical triplet $(\widehat{q},\widehat{p},\widehat{r})$, replacing the singular quantum space $\mathcal{S}$ of Heisenberg quantum theory by a finite quantum system space $\widehat{\mathcal{S}}$. We now do the same for the canonical pair $(t,-E)$. For $\hbar$ universality we take the action operator of time space to be identical with the action operator $\widehat{h}=\hbar\widehat{r}$ of system space. We write $\widehat{\mathcal{L}}:=\mathcal{L}(\widehat{q},\widehat{p},\widehat{t},\widehat{E})$ for the simple Lie algebra generated by the four flexed variables $\widehat{q},\widehat{p},\widehat{t},\widehat{E}$ subject to flexed commutation relations developed next.

The common action variable $\widehat{h}=\widehat{q}\times\widehat{p}=\widehat{E}\times\widehat{t}$ already couples the coordinate and momentum variables and couples the time and energy variables. The least orthogonal Lie algebra lg SO(n) that includes the sub-algebras $\mathcal{L}(\widehat{q},\widehat{p})$ and $\mathcal{L}(\widehat{t},\widehat{E})$ is apparently $\mathcal{L}(q,p,t,E)\sim$ lg SO$(2+\frac{\sigma}{2},2-\frac{\sigma}{2})$ for suitable signature σ. We must therefore include a boost $\widehat{b}\sim\widehat{q}\times\widehat{t}\sim\widehat{p}\times\widehat{E}\in\mathcal{L}(\widehat{q},\widehat{p},\widehat{t},\widehat{E})$ that couples the coordinate and time variables, and also couples the momentum and energy variables. By suitable conventions we shall associate the six operators $\widehat{q},\widehat{p},\widehat{t},\widehat{E},\widehat{r},\widehat{b}$ with multiples of the six infinitesimal generators $L^{ij}=-L^{ji}$ of SO$(3,1)$ with metric tensor

$$g=-1^1\otimes 1^1+1^2\otimes 1^2+1^3\otimes 1^3+1^4\otimes 1^4 \tag{38}$$

and commutation relations

$$L^{ij}\times L^{kl}=\frac{1}{2}[L^{ik}g^{jl}-L^{il}g^{jk}-L^{jk}g^{il}+L^{jl}g^{ik}]. \tag{39}$$

Up to sign, each of these commutation relations has one of the three typical forms

$$\begin{array}{rcl} L_{13}\times L_{13} &=& 0,\\ L_{13}\times L_{14} &=& L_{34},\\ L_{24}\times L_{13} &=& 0 \end{array} \tag{40}$$

depending on whether the two index pairs $\{i,j\}$ and $\{k,l\}$ agree in 2, 1 or 0 indices. Let us arbitrarily identify the 1 and 2 axes as respectively time-like and space-like, and the 3 and 4 axes as respectively coordinate-like and momentum-like. We introduce quantum constants $\hbar^{jk}$ as scale factors and relabel the six infinitesimal generators $\widehat{q},\widehat{p},\widehat{t},\widehat{E},\widehat{b},\widehat{r}$ as $\widehat{q}_{ij}$ according to

$$\begin{array}{ccccccccc} \widehat{b} &=& \widehat{q}_{12} &:=& \hbar^{12}L_{12}, & \widehat{q} &=& \widehat{q}_{23} &:=& \hbar^{23}L_{23},\\ \widehat{p} &=& \widehat{q}_{24} &:=& \hbar^{24}L_{24}, & \widehat{t} &=& \widehat{q}_{13} &:=& \hbar^{13}L_{13},\\ \widehat{E} &=& \widehat{q}_{14} &:=& \hbar^{14}L_{14}, & \widehat{r} &=& \widehat{q}_{34} &:=& \hbar^{34}L_{34}. \end{array} \tag{41}$$

Thus the constant $\hbar^{jk}$ is the quantum of the variable q^{jk}. In particular we write $\hbar_{13} =: \mathsf{T}$ for the time quantum or *chronon* and $\hbar_{14} =: \mathsf{E}$ for the energy quantum or *ergon*. This leads to 15 kinematic commutation relations

$$\begin{array}{rclrclrclrcl} \widehat{q}\times\widehat{p} &=& +\hbar i\widehat{r}, & \widehat{r}\times\widehat{q} &=& +\hbar_{rq}i\widehat{p}, & \widehat{p}\times\widehat{r} &=& +\hbar_{pr}i\widehat{q}, & \widehat{t}\times\widehat{E} &=& -\hbar i\widehat{r}, \\ \widehat{r}\times\widehat{t} &=& -\hbar_{rt}i\widehat{E}, & \widehat{q}\times\widehat{t} &=& +\hbar_{qt}i\widehat{b}, & \widehat{E}\times\widehat{r} &=& -\hbar_{Er}i\widehat{t}, & \widehat{E}\times\widehat{q} &=& 0, \\ \widehat{E}\times\widehat{p} &=& +\hbar_{Ep}i\widehat{b}, & \widehat{b}\times\widehat{E} &=& +\hbar_{bE}i\widehat{p}, & \widehat{b}\times\widehat{p} &=& +\hbar_{bp}i\widehat{E}, & \widehat{b}\times\widehat{q} &=& +\hbar_{bq}i\widehat{t}, \\ \widehat{b}\times\widehat{t} &=& +\hbar_{bt}i\widehat{q}, & \widehat{q}\times\widehat{E} &=& 0, & \widehat{p}\times\widehat{t} &=& 0 & & & \end{array} \tag{42}$$

with corresponding quantum constants [MORE to be checked and fixed!]

$$\begin{array}{lcllcl} \hbar &=& \hbar_{23}\hbar_{24}/\hbar_{34} = \hbar_{13}\hbar_{14}/\hbar_{34}, & \hbar_{rq} &=& \hbar_{23}\hbar_{34}/\hbar_{24} = \hbar_{14}\hbar_{34}/\hbar_{13}, \\ \hbar_{pr} &=& \hbar_{24}\hbar_{34}/\hbar_{23}, & \hbar_{Ep} &=& \hbar_{13}\hbar_{23}/\hbar_{12} = \hbar_{14}\hbar_{24}/\hbar_{12}, \\ \hbar_{bE} &=& \hbar_{12}\hbar_{14}/\hbar_{24} = \hbar_{12}\hbar_{13}/\hbar_{23}, & \hbar_{bp} &=& \hbar_{12}\hbar_{24}/\hbar_{14}, \\ \hbar_{bq} &=& \hbar_{12}\hbar_{23}/\hbar_{13}, & & & \end{array} \tag{43}$$

The relations (42) uniquely define a Lie algebra $\mathcal{L}(\widehat{\mathcal{S}}, \widehat{\mathcal{T}}) = \lg \mathrm{SO}(2+\frac{\sigma}{2}, 2-\frac{\sigma}{2})$ with associated dimension 4 and signature σ, but not a representation of that algebra. By Schur's Lemma, we single out an irreducible representation by fixing the values, for example, of the independent invariants $\mathrm{Tr}\ L^2, \mathrm{Tr}\ L^3, \mathrm{Tr}\ L^4$, where L is the matrix (L^j_k) of matrices, the powers shown are matrix powers, and the trace Tr is a diagonal sum over the external indices $j = k$ enumerating these matrices, and not over the internal indices enumerating their matrix elements.

The variables $\widehat{q}_{23} \sim \widehat{q}$, $\widehat{q}_{24} \sim \widehat{E}$ form a maximal commuting set among the flexed generators $\widehat{q}_{ij}$. We may therefore represent these operators on wave-functions of the form $\psi(\widehat{q}, \widehat{E})$. Similarly $\psi(\widehat{p}, \widehat{t})$ is a permissible representation. Since t and q do not commute, we cannot assume flexed oscillator wave-functions of the form $\psi(\widehat{q}, \widehat{t})$; nor of the form $\psi(\widehat{p}, \widehat{E})$.

In the singular limit the Heisenberg equation with time-independent Hamiltonian is invariant under time translation $t \to t + \Delta t$ and its infinitesimal generator E. There is no possibility of time-translation invariance in a regular theory since the spectrum of time is bounded, $|t| \leq N\hbar < \infty$. The eigenspaces for different values of t even have different dimensionalities. In the vicinity of $t = 0$ this variation with t is slow, and it disappears in the singular limit. These are the middle times of the theory.

We can however, when appropriate, require invariance under $\widehat{E}$. We take $\widehat{E}$-invariance to be the finite correspondent of the singular quantum E-invariance. In the singular quantum limit $\widehat{r} \to 1$ and $\widehat{E}$-invariance indeed approaches the usual time-translation invariance, though non-uniformly.

5.3 Flexing the commutation relations

In the singular theory dynamical operators at different times have commutators that are defined by equal-time commutators and the dynamical development. The finite correspondent of the singular equal-time algebra is here $\mathcal{A}(\mathcal{S})$. We are to construct the finite correspondent of the singular dynamical development $U = \exp(-iHt/\hbar)$.

The finite dynamical development respects the equal time commutation relations in the finite quantum theory when the finite Hamiltonian is invariant under $\widehat{E}$ and $\widehat{t}$.

Proof: The proof in the finite theory parallels the proof in the singular limit. In the singular quantum theory the development respects the equal time commutation relations when the Hamiltonian is invariant under E, t invariance being tautologous (t is central). In the finite theory, $\triangle\widehat{E} = \triangle\widehat{H}$ on the dynamical algebra. When $\widehat{H}$ commutes with $\widehat{t}$ and $\widehat{E}$, it also commutes with all of $\mathcal{A}(\mathcal{T})$. Then $\widehat{H}$ induces an automorphism of $\mathcal{A}(\mathcal{S})$, and one of $\mathcal{A}(\mathcal{S}) \otimes \mathcal{A}(\mathcal{T})$ that leaves the coordinates in $\mathcal{A}(\mathcal{T})$ invariant.

5.4 Flexing the dynamics

The finite dynamical equation found by flexing (35) is

$$[\widehat{E} - \widehat{H}] \times q =: \triangle(\widehat{E} - \widehat{H}) \cdot q = 0. \tag{44}$$

This resembles familiar dynamical equations if $\widehat{H}$ commutes with quantum time $\widehat{t} \sim \widehat{q}_{13}$ [by (41)] as the singular H commutes with singular time t. $\widehat{H}$ may contain any power of the variable $\widehat{p} \sim \widehat{q}_{24}$, including p^2, corresponding to the kinetic energy of the usual oscillator, and still commute with $\widehat{t}$. The quadratic term $\widehat{q}^2 \sim (\widehat{q}_{23})^2$ corresponding to the potential energy of the usual oscillator is forbidden by $\widehat{t}$- invariance, however, unless it is accompanied by a compensating term $\widehat{b}^2 \sim (\widehat{q}_{12})^2$ with the same scalar coefficient. Since the boost vanishes in the singular quantum limit this extra Hamiltonian term does not grossly violate experiment and may have experimental consequences.

In any case the finite dynamical equation (44) does not have the orthogonal symmetry $\mathrm{SO}(2+\frac{\omega}{2}, 2-\frac{\omega}{2})$ of the dynamical algebra. There exists a simple dynamics with this symmetry, for example that based on a quadratic Casimir operator

$$Q := C L^{\mu\nu} L_{\mu\nu} \tag{45}$$

with dynamical equation

$$\triangle Q \cdot V := C \triangle(L^{\mu\nu} L_{\mu\nu}) \cdot V = o \tag{46}$$

6 Conclusion

We suggest that all the infinities of present physics result from one or another flattening. Since quantum theory began as a regularization procedure of Planck, it is rather widely accepted that further regularization of present quantum physics calls for further quantization, but what to quantize and how to quantize it remains at least a bit unclear. If we regard quantization as a special case of group regularization, a path becomes clearer. It is marked by radicals ripe for annihilation. Group regularization of the linear harmonic oscillator results in a finite quantum theory with three quantum constants $\hbar, \hbar', \hbar''$ instead of the usual one. The stationary finite quantum oscillator is algebraically isomorphic to a dipole rotator with $N = l(l+1) \sim 1/(\hbar'\hbar'') \gg 1$ states and bounded Hamiltonian $H = A(L_1)^2 + B(L_2)^2$. Its position and momentum variables are quantized with uniformly spaced bounded finite spectra and supposedly universal quanta of position and momentum. For fixed quantum constants and large $N \gg 1$ there are three broad classes of finite oscillator, soft, medium, and hard. The field oscillators responsible for infra-red and ultraviolet divergences are soft and hard respectively. Medium oscillators have $\sim \sqrt{N}$ low-lying states having nearly the same zero-point energy and level spacing as the quantum oscillator and nearly obeying the Heisenberg uncertainty principle and the equipartition principle. The corresponding rotators are nearly polarized along the z axis with $L_3 \sim \pm l$.

The soft and hard oscillators have infinitesimal 0-point energy, and grossly violate both equipartition and the Heisenberg uncertainty relation. They do not resemble the quantum oscillator at all. Their low-lying energy states correspond to rotators with $L_1 \sim 0$ or $L_2 \sim 0$ instead of $L_3 \sim \pm l$. Soft oscillators have frozen momentum $\widehat{p} \approx 0$ because their maximum potential energy is too small to excite one quantum of kinetic energy. Hard oscillators have frozen position $\widehat{q} \approx 0$ because their maximum kinetic energy is too small to excite one quantum of potential energy.

The finite quantum theory affects cosmology. The zero-point energy of a physical oscillator presumably contributes to its gravitational field. It will be interesting to estimate the zero-point contribution to astronomical gravitational fields. For a consistent estimate we must regularize the space-time operators x^μ, ∂_μ as well as the canonical field variables q, p, since both algebras have the same instability. This changes not only the structure of the individual oscillators, as considered here, but also the number and distribution of the oscillators. We take a crucial first step in this direction by quantizing time and energy in this note, and leave the next step for further study.

The finite quantum theory modifies low- and high-energy physics. Because the low-lying energy levels of medium oscillators have nearly uniform spacing, the energy of two excitations is but slightly less than the sum of their separate energies. The corresponding quanta nearly do not interact, and the small interaction that they have is attractive. For soft or hard oscillators, the energy level varies quadratically with the energy quantum number. The energy of two quanta of oscillation is twice the sum of their separate energies, for example. The corresponding quanta have a repulsive interaction of great strength; the interaction energy is equal to the total energy of the separate quanta. Thus the simplest regularization leads to interactions between the previously uncoupled excitation quanta of the oscillator, weakly attractive for medium quanta, strongly repulsive for soft or hard quanta.

Like Dirac's theory of the "anomalous" magnetic moment of the relativistic electron, these extreme-energy effects depend on factor ordering. They can be adjusted to fit the data by re-ordering factors and so are not crucial tests of the theory. The theory of a more physical system will be necessary for that.

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