General quantization

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Abstract

Segal's hypothesis that physical theories drift toward simple groups follows from a general quantum principle and suggests a general quantization process. I general-quantize the scalar meson field in Minkowski space-time to illustrate the process. The result is a finite quantum field theory over a quantum space-time with higher symmetry than the singular theory. Multiple quantification connects the levels of the theory.

1 Quantization as regularization

Quantum theory began with *ad hoc* regularization prescriptions of Planck and Bohr to fit the weird behavior of the electromagnetic field and the nuclear atom and to handle infinities that blocked earlier theories. In 1924 Heisenberg discovered that one small change in algebra did both naturally. In the early 1930's he suggested extending his algebraic method to space-time, to regularize field theory, inspiring the pioneering quantum space-time of Snyder [43]. Dirac's historic quantization program for gravity also eliminated absolute space-time points from the quantum theory of gravity, leading Bergmann too to say that the world point itself possesses no physical reality [5, 6].

For many the infinities that still haunt physics cry for further and deeper quantization, but there has been little agreement on exactly what and how far to quantize. According to Segal canonical quantization continued a drift of physical theory toward simple groups that special relativization began. He proposed on Darwinian grounds that further quantization should lead to simple groups [32]. Vilela Mendes initiated the work in that direction [37].

Each non-simplicity of the operational algebra arises from an idol of the theory in the sense of Bacon [3]. An idol is a false absolute in which we believe beyond the experimental evidence, a construct that we assume to act but not to react. Today it may be more practical to topple these idols than to work around them.

Invariant subgroups and Lie algebra ideals correspond to idols and force us to infinite-dimensional representations. Simple Lie algebras have enough finite-dimensional representations. We relativize the absolute and regularize the theory by simplifying the Lie algebra: slightly changing its structural tensor so that the Lie algebra becomes

*School of Physics, Georgia Institute of Technology, Atlanta, Georgia. df4@mail.gatech.edu. This updates part of a talk given at Glafka 2005, Athens. quant-ph/0601002 simple, or at least simpler. An arbitrarily small homotopy of the structure tensor often suffices for simplicity. Lie algebra simplification is a key step in special and general relativization, canonical quantization and general quantization.

Physics has several levels and canonical quantization simplifies only the highest-level Lie algebra, and that not all the way to simplicity and finiteness. General quantization extrapolates canonical quantization in both respects. It simplifies the Lie algebras on all the known levels of a physical theory, and it simplifies them all the way to simplicity and finiteness. It does this by small changes in the structure tensor so that hopefully it makes only small changes in experimental predictions near the group identity, in the correspondence domain.

For exercise and illustration we general-quantize the scalar meson quantum field here. A first-level quantization of space-time or the ether resolves it into an aggregate of many identical finite quantum elements, chronons. A second-level general quantization of the field resolves the field history into aggregates of chronons. The vacuum mode-vector $|\mathrm{vac}\rangle$ represents the ambient mode of the ether. General quantization infers candidate structures and symmetries for the ether and its elements from the structure and symmetry of the present-day vacuum by a routine heuristic procedure based on correspondence, simplicity, and symmetry.

2 Less is different too

More is different [1]; different from less, one understands. When we pass from few systems to many we encounter self-organization, with more structure, less symmetry.

It follows that less is different too; different from more. When we pass from many systems to few, as from the ether to the sub-ether, we expect to lose organization, gain symmetry, and lose structure. General quantization accords with this expectation. Discretization, however, and bottom-up reconstructions of the sub-ether like vortex, network, string, and loop models, enrich its structure, reduce its symmetry, and increase its singularity.

Simple Lie algebras have quite special dimensions. There is no simple Lie algebra of dimension 2, for example, and yet there is the stable two-dimensional compound (= non-semisimple) Lie algebra $W_2(q, p)$ with $qp - pq = q$. Therefore general quantization often requires us to introduce new dynamical variables into the Lie algebra, called regulators, to raise its dimensionality to that of a simple Lie algebra. Then we must also invoke a self-organization, a spontaneous symmetry-breaking, to freeze out these new variables and recover the singular theory in a limited correspondence domain. Special relativization and canonical quantization introduced no regulators but further simplifications will.

2.1 Theory drift

Early on Von Neumann and Wigner noted that some important evolutions are small homotopies of the algebra. Segal suggested the *simplicity principle*: Theories drift toward (group) simplicity. His reason is essentially Darwinian. Our experiments are disturbed by the many uncontrolled quantum variables of the experimenter and the medium, so our measurement of the structure tensor must err. To survive a physical theory should be stable against small errors in the structure tensor. Simple Lie algebras are stable Lie algebras.

This proposal rests on dubious implicit assumptions about the domain of possibilities. For instance, groups that are stable in the domain of Lie groups are unstable within the larger domain of quantum groups or non-associative products. The group-stability criterion might produce some useful theories, but it might also exclude some. Moreover there are a great many stable algebras that are not simple.

We infer the simplicity principle from the *general quantum principle*: Any isolated system is a quantum system, with unlimited superposition. Therefore the commutator algebra of its operational algebra is is the special orthogonal algebra of its io vector space (§4), which is simple. A system whose algebra is not yet simple is one we have yet to resolve into its quantum constituents, possibly because of strong binding or low resolution.

There are encouraging signs that when the algebra becomes simple the theory becomes finite; that infinities today result from departures from algebra simplicity, vestiges of classical physics that must be quantized.

The simplicity principle provides the kind of general understanding of the development of physics that Darwin's theory of evolution and Wegener's theory of continental drift supply for biology and geology. It does not predict the development but suggests several possibilities for experiment to choose among. It produces a phenomenological theory, not a "fundamental theory."

One can illustrate such regularization-by-simplification with the same elementary example as before (§1). The quantum linear harmonic oscillator has compound and singular Lie algebra and infinite-dimensional mode-vector space. Its basic coordinate and momentum operators diverge on most of its mode vectors. Segal stabilized this algebra by simplifying it, probably to $d\mathrm{SO}(2,1)$, though he did not state the signature explicitly. This has an irreducible representation $R(l)\mathrm{SO}(3)$ of finite dimension $2l+1$ for any finite quantum number $2l = 0, 1, 2, \ldots$. For all our finite experiments can tell us, a sufficiently high-but-finite-dimensional matrix representation works at least as well as the infinite-dimensional singular limit. Heisenberg's historic choice of the singular compound group $H(1)$ over the regular simple one $\mathrm{SO}(2,1)$, and Einstein's choice of the compound Poincaré group over a simple de Sitter group, natural as they were, incorporated typical idol-formations of the kind that Bacon described. The algebra simplifications described here regularize the theories as well as stabilizing them in some degree. For another example where a well-chosen homotopy replaces infinite-dimensional representations of a compound group by finite-dimensional ones of a simple group see [23, 24].

In general, the irreducible representations of compound algebras useful in physics are nearly unique but contain serious infinities, while representations of inearby simple algebras are numerous but finite. It seems plausible that some of these nearby finite-dimensional algebras suffice for present physics at least as well as the present infinite-dimensional ones.

If the operational algebra of a system is not simple, it omits some quantum degrees of freedom. As these are found and excited the theory will correspondingly simplify its Lie algebra. Theory currently drifts toward simplicity because we are increasing the resolution of our analysis of nature into quantum elements. This is only one of the currents at work in the drift of physical theory.

2.2 The oldest game in town

The deep changes in the structure of successful physical theories since 1900 — special and general relativity, quantum theory, gauge theory — introduced simplifications, a mode of theory change that lends itself to mathematical study. The well-posed direct problem is how present theories contract to past ones [21]. Segal worked on the ill-posed inverse problem that concerns us here [32]. How should present theories expand into future ones?

Vilela Mendes [37] seems to have been the first to apply the stability principle of Segal to construct new quantum physical theories. He noted that to simplify most Lie algebras one must first introduce new variables and then invoke crystallization to freeze them out in the vacuum. He drew on the mathematical theory of stable (rigid) algebraic structures [19], which in turn may have been influenced by Segal's proposal.

People have since simplified the stationary theory of a quantum harmonic oscillator [25, 8, 2, 36, 4, 33] and some its canonical dynamics [4, 34]. Madore's "fuzzy spheres" include the Segal simplification [32] of the Heisenberg algebra $d\mathrm{H}(1)$ with one coordinate and one momentum as a special case [27, 28]. Golden has simplified a current algebra in the manner of Vilela Mendes [20].

The present simplification program stems from the stabilzation program of Segal [32]. (§sec:STABLE) It extends the stabilization of space-time by Vilela Mendes [37] to the higher level of dynamics.

3 General quantization

Singular theories are based on a Lie algebra $\mathcal{L}(0)$ that is compound, not semisimple, and on a representation thereof — call it $R(0)\mathcal{L}(0)$ — that is not finite-dimensional. For the canonical algebra $d\mathrm{H}(1)$, $qp - pq = ihbar$, ..., the representation $R(0)$ is uniquely determined by unitarity, irreducibility, and the value of one quantum constant $\hbar$ and is infinite-dimensional.

A *central invariant* is an algebraic combination of Lie algebra elements that is central in every representation, and is therefore a c number in every irreducible representation. For any Lie algebra $\mathcal{L}$ and any representation R of $\mathcal{L}$, the Casimir invariant C_n of $R\mathcal{L}$, the coefficient of z^n in the characteristic polynomial

$$C(z) = \det(L - z\mathbf{1}) = \sum C_n z^n \quad \mathrm{for} L \in R\mathcal{L} \tag{1}$$

is a central invariant. Planck's constant in the form $i\hbar$ is the value of the central invariant $r = i\hbar$ for $d\mathrm{H}(1)$.

It is convenient to introduce dimensional constants δq_n to bring the generators q_n of $\mathcal{L}$ to a standard dimensionless form L_n whose spectrum has unit spacing. Then the C_n have integer eigenvalues c_n, *quantum numbers* that define a representation algebra $R(\mathbf{c})\mathcal{L}$.

The algebra does not define its own physical meaning. One must give its elements physical meaning. If we double the value of $\hbar$ we change physical predictions but do not change the algebra, up to isomorphism. We may form a distinguished *physical basis* $\mathbf{B}$ within the Lie algebra $\mathcal{L}$, of physical variables q_n defined by how we measure them in standard units, not by their algebraic relations; for example, q_1 = position,

q_2 = momentum, ... *Quantum constants* $\mathbf{h} = \{\delta q_n\}$, including $\hbar$, then define the representation $\mathbf{B}(\mathbf{h})$ of the preferred basis within the representation $R(\mathbf{c})\mathcal{L}$.

To quantize a singular theory in the present general sense we:

1. Warp its Lie algebras to simple ones with as few new variables as possible.
2. Choose representations that correspond with the singular theory in the experimental domain.

The correspondence principle provides experimental meanings for some of the variables.

The "space-time code" [12, 15] built quantum space-time from the bottom up. Vilela Mendes [37, 40] and the present work build quantum space-time from the top down. The c space-time continuum arises from STiME in a singular limit of an organized mode of an underlying complex system, the ether, which determines no rest frame. STiME splits into the usual fragments — space-time, the complex plane, and momentum-energy — only relative to the ether.

The q space-time has a basic kinematic symmetry between space-time and energy-momentum variables like that postulated by Born and co-workers in their reciprocity theory [7], except that now it extends to i as well, whose simplification $\widehat{\imath}$ couples p and q, E and t. This unification of time and energy violates common sense even more than the unification of time and space, though this does not mean it is right. The ether condensation breaks this symmetry.

The present singular theory uses several singular algebras. For example, classical mechanics has both a commutative algebra of phase-space coordinates and a Lie algebra of phase space coordinates with the Poisson Bracket as product. Classical space-time has a commutative algebra of coordinates and a Lie algebra of vector fields with the Lie Bracket as product. In such cases economy prefers a quantization that deduces both singular algebras as singular limiting cases of one more regular algebra, as did canonical quantization.

Shall we follow the orthogonal, unitary, or symplectic line of simple algebras? We work with huge dimensionality, so the exceptional algebras do not come in. Experiment does not yet clearly decide our choice.

I exclude the symplectic line because it lacks a well-behaved quantification theory. The best-behaved quantification is that of the Clifford-Wilczek statistics of the real orthogonal D line, which enjoys the "family property" [45]. We take the D line here. Baugh takes the A line [4]. It is too soon to say which agrees better with experiment.

3.1 Terminology

Some terms:

A † space V is a vector space provided with an involutory antilinear possibly indefinite anti-automorphism $\dagger : V \to V^{\mathrm{D}}$, the dual space. The † represents total time reversal [15, 31]. In a quantum theory, positive-signature unit vectors $|\psi\rangle \in V$, $\psi^\dagger(\psi) = +1$, represent input modes; positive-signature unit dual vectors $\langle\phi| \in V^{\mathrm{D}}$ express output modes; the transition amplitude between them is $A = \langle\phi|\psi\rangle$.

We deal with both abstract and operational algebras or groups. An operational algebra is an algebra with an operational interpretation. The interpretation may be expressed by assigning names like momentum or charge to elements, which define how they are executed in the laboratory. It suffices to do this for a basis. Similarly for groups. Stretching some of the preferred basis elements does not change the abstract algebra or the representation algebra but it changes the physical interpretation, and therefore the operational algebra.

A *stable* † Lie algebra is one whose Lie product $\mathsf{X} : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}, a \otimes b \mapsto [a, b]$ is isomorphic to all the Lie products $\mathsf{X}' \in N(\mathsf{X})$, a neighborhood of X that are compatible with the same †. (Segal's stability ignores †.) [32]. Because a non-singular † is stable, we need not modify it.

Some concepts have been formulated and named several times. A stable algebra is variously and synonymously said to be robust, rigid, regular, or generic. Conversely an unstable algebra is said to be fragile, elastic, singular, or special. The operational algebras in a neighborhood differ slightly in interpretation, assuming that the distinguished basis is unchanged, and so for physical purposes Segal's term "stable" is better than "rigid."

We can usually ignore the difference between the simple and semi-simple here. A direct sum is an incoherent mixture, and we see only one system at a time. One well-chosen maximal measurement will reduce a semi-simple operational algebra to one of its simple "superselected" terms for all subsequent measurements.

A homotopy $A_0 \hookrightarrow A_1$ from one † algebra A_0 to another A_1 (possibly Lie) on the same † space A, each with its own product X_0 and X_1, is a continuous function $\mathsf{X} : A \otimes A \times I \to A$, where $I = [s_0, s_1] \subset \mathbb{R}$ is an interval, such that $\mathsf{X}(a, a', s_0) = a\mathsf{X}_0 a'$, $\mathsf{X}(a, a', s_1) = a\mathsf{X}_1 a'$, and each $\mathsf{X}(a, a', s) = a\mathsf{X}_s a'$ is a † algebra product for all $s \in I$. Usually $s_0 = 0$.

A *simplification* is a homotopy $A \hookrightarrow A(s)$ from a non-simple algebra $A = A(0)$ to a simpler algebra $A(s)$ (say, with smaller ideals or nilradical), with homotopy parameter $s \in [0, s_1] \subset \mathbb{R}$. As an example Segal simplifies a canonical Lie algebra $\mathrm{Lie}(q, p, i)$ to a Lie algebra of three generating angular-momentum-like variables $\mathrm{Lie}(\widehat{q}, \widehat{p}, \widehat{r})$, replacing the central i with the non-central r. His homotopy transformation $L \times [0, s_1] \to L$ depends quadratically on s. The kind of homotopy that Inönü and Wigner called a contraction [21] is a special case that we call a linear contraction. Non-linear contractions are necessary as well [42, 31]. Linear contractions sufficed to contract special relativity to Galilean relativity and quantum theory to classical mechanics. To recover classical mechanics from canonical quantum mechanics requires a quadratic contraction. The regularizations of bosonic statistics and of space-time structure. are inverses of contractions in the more general non-linear sense.

3.2 Quantum constants

General quantization usually introduces new regulation operators or *regulators* q_n and several kinds of physical constant:

1. Signatures defining the simplified Lie product $\widehat{\mathsf{X}}$.
2. Regulation constants or *regulants* $\overline{q}_n$, expectation values of regulants q_n in the ambient ether, setting physical scales.
3. *Quantum numbers* $\mathbf{c}$ defining a representation $R(\mathbf{c}) : \mathcal{L} \to A(\mathbf{c})$ of the simplified Lie algebra.
4. *Quantum constants* δq_n defining the spectral intervals of physical operators $q_n \in A(\mathbf{c})$.

The regulants $\overline{q}_n$ are typically both spectral maxima and ambient values of regulators,

$$\max |q_n| =< \mathrm{vac}|q_n|\mathrm{vac} >:= \overline{q}_n. \tag{2}$$

We simplify the canonical relation $pq - qp = -i\hbar$ to the cyclic form

$$\widehat{p}\widehat{q} - \widehat{q}\widehat{p} = \frac{\delta p\, \delta q}{\delta r}\widehat{r}, \quad \& \text{ cyc} \tag{3}$$

on dimensional grounds. The operator that freezes to $-i\hbar$ in the singular theory is clearly

$$\widehat{\imath\hbar} = \frac{\delta p\, \delta q}{\delta r}\widehat{r} := N\delta p\, \delta q. \tag{4}$$

where the integer N is the maximum eigenvalue of $|\widehat{r}|$ as a multiple of its quantum δr.

Canonical quantization and special-relativization introduced scale or quantum constants but no regulators. Subsequent simplifications have both [32, 37, 4, 33, 34].

Theories are c or q as their dynamical variables all commute or not. Then q theories divide into q/c and q/q as their time is commutative or not [15]. We formulate a q/q physics here

The two main ways to formulate a c dynamical theory, Hamiltonian and Lagrangian, have q/ and q/q correspondents. When the Hamiltonian theory is singular, the Lagrangian theory is even more so, because the system has many more history modes than single-time modes. I simplify the Lagrangian theory.

In a q/q theory, the algebras of all levels within the theory are non-commutative. To regularize such hierarchic theories we must regularize all their constituent algebras and the algebraic relations between levels. For this we use an algebraic concept of quantification (§4) or statistics.

4 Quantification

The passage from a one-system theory to a many-system theory is a general process aptly named quantification by the Scottish logician William Hamilton (1788- 1856). It is not a quantization and is much older.

The operations of system creation and annihilation can be represented as withdrawals and deposits from a reservoir of like systems. The one-quantum theory keeps the reservoir off-stage and represents the io actions not by operators but only by vectors, halves of operators. Quantification brings the reservoir on-stage and represents these io actions by operators, not mode vectors. The notion that experiments on a single quantum can tell us the operational algebra of a many-quantum system is a relic of c physics, where bodies are made of atoms and the body state space is the Cartesian product of many atom state spaces, but it might still be right, and has worked amazingly in the q theory, with necessary changes.

The earliest quantum physicists naively took for granted that the many-quantum mode space is the tensor algebra over V, Tensor$V$. This is Maxwell-Boltzmann statistics, the quantification for fictitious quanta that we can call maxwellons. Then Bose realized that the io operators generating the quantified algebra must obey significant commutation relations. For example the bosonic and fermionic quantifications Σ_σ are based on the algebraic relation

$$b^\dagger a = \sigma a b^\dagger + \langle b|a\rangle \tag{5}$$

for $a, b \in V$. We confine ourselves here to the algebras with $\sigma = +$ (bosonic), $\sigma = -$ (fermionic), $\sigma = 0$ (maxwellonic) and their simplifications and iterations.

The Lie algebra (5) for $\sigma = +$ is a singular canonical algebra $d\mathrm{H}(D)$. We simplify it to a regular orthogonal algebra $d\mathrm{SO}(2D+2) \hookrightarrow d\mathrm{H}(2D)$.

Standard quantum theory forbids the superposition of bras and kets and bars total time reversal from the operational algebra. This rule seems to be a c vestige in the present quantum theory. We break it and form the direct sum of the two io spaces of bras V and kets V^{D} [31], calling this the io (vector) space

$$W = V \oplus V^{\mathrm{D}}. \tag{6}$$

By a quantification Σ I mean a construction that sends each one-quantum io space W_1 to a many-quantum io space $W = \Sigma W_1$ depending on

- the structure X of a † Lie algebra $\mathcal{L}(W_1, \mathsf{X})$ on W_1,
- an irreducible representation $R : \mathcal{L}(W_1, \mathsf{X}) \to A$, and
- a vacuum projector $P(\mathrm{vac}) \in A$

The quantified io vector space is then defined as

$$W = \Sigma W_1 = R\mathcal{L}(W_1, \mathsf{X})\ P(\mathrm{vac}). \tag{7}$$

If we diagonalize $P(\mathrm{vac})$, all the operators $\Sigma W_1\ P(\mathrm{vac})$ are matrices with only one non-zero column which represents a ket.

To define a quantification we must give not only the Lie algebra but must also give values $\mathbf{c} = \{c_n\}$ for its central (Casimir) invariants $\mathbf{C}$ to define an irreducible representation $R(\mathbf{c}) : \mathcal{L} \to A$, the endomorphism algebra of the † space W.

We regard Σ_{X} for any Lie algebra $\mathcal{L} = (W, \mathsf{X})$ as a generalized statistics or quantification for quanta that we can call "$\mathcal{L}$-ons." Bosons and fermions result from Bose and Fermi (graded Lie) algebras respectively, which are canonical and Clifford algebras respectively. In the fermion case the input and output vectors in W are required to be null vectors in the Clifford algebra. This is a singular condition that has already been regularized for other reasons [14, 45, 15]. We regularize the bosonic statistics in §6.3.

Since the classical continuum is singular, we regard all our Lie algebras as ultimately statistical.

Dynamics has a hierarchy of at least five algebras (6). In formal logic such hierarchies are handled with quantifiers. In q/c physics the lower level c quantification is handled informally and intuitively, and the higher q level quantification is constructed from the lower algebraically as in §4. In q/q physics we must handle all quantifications algebraically.

Quantification deceptively resembles quantization in more than spelling. Both adduce commutation relations, and they may even end up with the same algebra. Nevertheless they are conceptual opposites and if they come to the same place, they arrive there from opposite sides. Quantification sets out from a one-quantum theory. Quantization set out from a classical theory, which is a many-quantum system seen under low resolution and with many degrees of freedom frozen out. For extremely linear systems like Maxwell's, the two starting points may have similar-looking variables but the operational meanings of those variables are as different as c and q.

5 Finiteness and stability

Simple Lie algebras seem to result in finite (= convergent) theories. We begin to explore this delicate question here. Compact simple Lie algebras have complete sets of finite-dimensional representations supporting finite-dimensional quantum theories with no room for infinities. The simple algebras with indefinite metric required for physics have problematic infinite-dimensional irreducible unitary representations besides the good finite-dimensional ones. We hypothesize that we can approximate the older singular compound theory without these infinite-dimensional representations; this has been the case for the Lorentz group, for example, and it is consistent with analytic continuation from the compact case of positive definite †.

Regularity also divides mechanical theories with singular Hessian determinants from those with regular Hessians. Indeed, all singularities that depend on some variable determinant miraculously vanishing are non-robust, non-generic, unstable by that fact, and are eliminated by general quantization.

5.1 Regularization by simplification

The Lie algebraic products, or structure tensors, $\mathsf{X} : V \otimes V \to V$ admitted by a given vector space V, form a quadratic submanifold $\{\mathsf{X}\}$ in the linear space of tensors over V, defined by the Lie identities

$$\mathsf{X}(a \otimes b + b \otimes a) = 0, \quad \mathsf{X}^2(a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a) = 0. \tag{8}$$

The quotient of this manifold by the equivalence classes modulo Lie-algebra isomorphism is the moduli space of Lie algebras on V [11].

Any singular Lie algebra lies on the lower-dimensional boundary in $\{\mathsf{X}\}$ of a finite number of these classes. For example, the 6-dimensional Galilean algebra of rotations and boosts sits between the SO(4) algebras and the SO(3, 1) algebras. To simplify such a singular algebra we merely move its structure tensor off this boundary to a nearby simple algebra [32, 19, 37, 31]. The simple group approximates the compound one only near their common point of tangency, as a sphere approximates a tangent plane, in a correspondence domain whose size is set by a physical constant or constants new to the singular theory, and which must include the experiments that have been satisfactorily described by the singular theory. A simplified theory $\widehat{\Theta}$, by fitting its regulants into the error bars of the unsimplified theory Θ, inherits the operational semantics and past experimental validations of Θ, while still making radically new theoretical predictions about future experiments,

5.2 Regulators

If we introduce regulators we also need to explain how the unregulated singular theory could work as well as it does without them. Call the subspace of the regular mode-vector space where the regular theory agrees with the singular theory within experimental error, the *correspondence domain*. We hypothesize that self-organization produces a gap that freezes out the regulators in the correspondence domain, where the singular theory gives some good results. Self-organization is also responsible for the emergence of classical mechanics from quantum mechanics.

General quantization exposes a larger symmetry algebra, supposed to have been hidden in the past by self-organization, and able to manifest itself in the future under extreme conditions like ether melt-down. Carried far enough, general quantization converts a singular theory with a compound (= non-semisimple) algebra into a regular theory with a simple algebra [32]. This requires no change in the stable elements of a theory, only in the unstable elements, such as the classical theory of space-time.

Suppose that the simple Lie algebra is an orthogonal one $d\mathrm{SO}(N)$ (rather than unitary or symplectic). Then we can choose each simplified generating variable q to be a multiple of an appropriate dimensionless component $L^{\alpha}{}_{\beta}$ of an angular momentum in N dimensions, by a dimensional constant δq:

$$\widehat{q} = \delta q L^{\alpha}{}_{\beta}. \tag{9}$$

We adjust the spectral spacing of $L^{\alpha}{}_{\beta}$ to 1. Then the quantum of q is δq. To diagonalize an antisymmetric generator $L^{\alpha}{}_{\beta}$ requires adjoining a central i for the purpose. Then the generators are all quantized with uniformly spaced, bounded, discrete spectra. The maximum of the absolute values of the eigenvalues of $\widehat{q}_n$ we designate by $\mathrm{Max}\widehat{q}_n$.

These δ's generalize the quantum of action, $\delta A = \delta(E/\omega) = \hbar$, so we call them quanta of their variables. For example, simplification introduces quanta δx of position, δt of time, δp of momentum, and δE of energy, as well as the familiar quanta of charge and angular momentum.

The main singular algebra of q/c physics, the Heisenberg algebra $d\mathrm{H}(M)$ (for M spatial dimensions), whose radical includes $i\hbar$, has already been simplified for $M = 1$ [32, 37, 25, 8, 2, 4, 33, 36] and for $M > 1$, both unitarily [4] and orthogonally [33, 40].

6 A regular relativistic dynamics

Can the infinite-dimensional representations of the non-compact groups like the Poincaré group that are used in quantum physics today indeed be approximated by finite-dimensional representations of an approximating orthogonal group? In the non-compact cases the orthogonal groups have infinite-dimensional irreducible unitary representations as well as finite-dimensional orthogonal ones. The danger is that an infinite-dimensional representation is required for this approximation, with its native divergences.

A typical example: Consider a scalar quantum of mass m in a space-time of $3+1$ dimensions. One can approximate its singular Poincaré Lie algebra $d\mathrm{ISO}(3,1)$ with a regular de Sitter Lie algebra $d\mathrm{SO}(5,1) \to d\mathrm{ISO}(3{,}1)$. A scalar massive quantum in Minkowski space-time provides an infinite-dimensional unitary representation $R\, d\mathrm{ISO}(3,1)$ in use today. Can one approximate this useful infinite-dimensional representation of the singular algebra by a finite-dimensional representation of the regular algebra?

At least five major Lie algebras arise in such models:

Level	Space		Lie algebra
1	Space-time tangent space	$d\mathcal{X} = \{dx\}$	$d\mathrm{SO}(3,1)$
2	Space-time	$\mathcal{X} = \{x\}$	$\mathcal{L}_{\mathcal{X}}$
3	Field-value tangent space	$d\mathcal{F} = \{df\}$	$\mathcal{L}_{d\mathcal{F}}$
4	Field-value space	$\Phi = \{\phi\}$	$\mathcal{L}_{\Phi}$
5	Field history space	$\mathcal{F} = \{f\}$	$\mathcal{L}_{\mathcal{F}}$

The initial hierarchic structure is a lambda we assume, with space-time and field variable on the same level, and the final structure is simpler:

$$
\begin{array}{ccccccc}
 & & \mathcal{F} & & & & \widehat{\mathcal{F}} \\
 & \nearrow & & \nwarrow & & & \uparrow \\
 & \mathcal{X} & & \Phi & & \rightarrow & \widehat{\mathcal{X}} \\
\nearrow & & & & \nwarrow & & \uparrow \\
d\mathcal{X} & & & & d\Phi & & d\widehat{\mathcal{X}}
\end{array}
$$

The Lorentz algebra $\mathcal{L}_{d\mathcal{X}}$ is regular and for the scalar field $\mathcal{L}_{\Phi}$ and $\mathcal{L}_{d\mathcal{F}}$ are the one-dimensional Lie algebra. We regularize the remaining algebras here.

6.1 Regular space-time

We simplify space-time first, then the scalar field on that space-time. This is mainly an illustrative example chosen as training for the most interesting singularity, that of gauge theory and gravity, which suggests a quantum space-time that we take more seriously.

The usual space-time coordinates x^{μ} generate a compound commutative four-dimensional Lie algebra. There is no 4-dimensional simple Lie algebra. To make simplification possible without losing Lorentz invariance we adjoin the four differential operators ∂_{μ} and 1 as regulators, resulting in the compound Lie algebra $\mathrm{H}(4) = \mathrm{Lie}(x^{\mu}, \partial_{\mu}, 1)$ with standard commutation relations understood. This may also be the most economical way.

Now the irreducible unitary representation is essentially unique: The generators $x^{\mu}, \partial_{\mu}, 1$ act in the standard way on $\mathrm{L}^2(\mathcal{M}^4)$. This is also isomorphic to the diachronic pre-dynamical operational Lie algebra of a single scalar quantum particle in space-time. Statements about position in the abstract have been imbedded in statements about a quantum particle of unspecified dynamics, which we call "the probe." Inevitably this brings in statements about the momentum of the probe as well. This is but a partial regularization of space-time, neither regular nor simple.

$\mathrm{Lie}(x^{\mu}, \partial_{\mu}, 1)$ is also the Lie algebra $\Sigma_+ V(3,1)$ of a certain bosonic aggregate. The mode-space $V(3,1)$ of the individual boson is isomorphic to the tangent space $d\mathcal{M}^4$ to four-dimensional Minkowski space $\mathcal{M}^4$ at the origin but is not that space, being interpreted in a way that is non-standard for differential geometry. Its vectors are mode-vectors of a hypothetical quantum; the "minkowskion," let us call it. The classical space-time is now presented, ready for regularization, as a bosonic aggregate of minkowskions which has been reduced to a classical system by freezing out the momentum-energy variables, and centralizing ("superselecting") the coordinates x^{μ}, effectively restricting frames to the classical space-time coordinate basis $|x^{\mu}\rangle$. No quantum of space-time has entered yet, but quantum variables have. To take quantum space-time seriously one must eventually find a physical mechanism that freezes half the variables by self-organization (6.3).

Now we simplify fully. This calls for more regulators. We follow the D line and adjoin 6 Lorentz generators $L^{\mu}{}_{\nu} = -L_{\nu}{}^{\mu}$ to the present generators x^{μ}, ∂_{μ}, assuming a fixed background Minkowski metric $\dagger$ that interchanges vectors and dual vectors, raising and lowering indices. This expands the 9-dimensional canonical Lie algebra $d\mathrm{H}(4)$ to a still singular 15-dimensional Lie algebra $\mathrm{Lie}(x^{\mu}, \partial_{\mu}, L^{\nu}{}_{\mu}, 1)$ with the commutator $AB - BA$ as Lie product $[A, B]$ and with standard commutation relations (11) for these operators. This algebra can be simplified to a 15-dimensional orthogonal algebra $d\mathrm{SO}(6)$ of signature to be determined.

This simple space-time is more quantum than the Snyder space-time, which is not simple.

Notation: We label simplifications by a collective argument $\mathbf{q} = (\mathbf{h}, \mathbf{c})$ with $\mathbf{h} = \{\delta q_i\} = \delta\mathbf{q}$ consisting of quantum constants like $\hbar$ and $1/c$, and with $\mathbf{c} = \{c_n\}$ consisting of quantum numbers defining values of all central invariants (see §3). The passage to a singular limit we write as $\mathbf{q} \to \mathbf{q}_0$. We absorb factors of i to make the variables q_i anti-Hermitian for convenience. We may omit the circumflex that indicates simplification when it is redundant. The old indices $\mu, \nu = 0, 1, 2, 3$ label space-time or momentum-energy axes in the singular theory. Special constant index values X, Y label real and imaginary units in the complex plane of the singular limit. They distinguish space-time variables $L_{\mu \mathrm{X}}$ from momentum-energy variables $L_{\mu \mathrm{Y}}$ in the regular theory. Extended indices $\alpha, \beta = 0, 1, 2, 3, \mathrm{X}, \mathrm{Y}$ label axes in the orthogonal space that supports the regular group SO(5, 1). We may set $\hbar = c = 1$ since we hold them constant as $\mathbf{q} \to \mathbf{q}_0$. $\mathcal{X}(\mathbf{q}_0)$ is the singular quantum space and the associative algebra defined by the usual infinite-dimensional unitary representation of its Lie algebra

$$\mathcal{L}_{\mathcal{X}}(\mathbf{q}_0) := \mathrm{Lie}(x^\mu, p_\mu, L^\nu{}_\mu, i) \tag{10}$$

on the function space $\mathrm{L}^2(x^\mu)$. $\mathcal{L}_{\mathcal{X}}(\mathbf{q}_0)$ has the familiar singular structure

$$\begin{array}{llrcllcl} [x^\nu, x^\mu] = 0, & [x^\nu, p_\mu] = i\delta^\nu_\mu, & [x^\nu, L_{\mu\lambda}] & = & \delta^\nu_\mu x_\lambda - \delta^\nu_\lambda x_\mu, & [x^\mu, i] & = & 0, \\ & [p^\nu, p^\mu] = 0, & [p^\nu, L_{\mu\lambda}] & = & \delta^\nu_\mu p_\lambda - \delta^\nu_\lambda p_\mu, & [p^\mu, i] & = & 0, \\ & & [L^{\nu\mu}, L_{\lambda\kappa}] & = & \delta^{[\nu}_{[\lambda} L^{\mu]}{}_{\kappa]}, & [L_{\nu\mu}, i] & = & 0 \end{array} \tag{11}$$

The singular Lie algebra $\mathcal{L}_{\mathcal{X}}(\mathbf{q}_0)$ simplifies to a regular Lie algebra $\widehat{\mathcal{L}}_{\mathcal{X}}(\mathbf{q}) \sim d\mathrm{SO}(5,1)$ or $d\mathrm{SO}(3,3)$ as follows [37].

First the idol i melts down to the Lie element $\widehat{\imath} := \widehat{r}/\mathrm{Max}\,\widehat{r}$. We return to the singular theory by freezing $\widehat{\imath}$ at its maximum eigenvalue.

We choose the Minkowskian signature to postpone the problems of multiple timelike axes. Then the dimensionless infinitesimal orthogonal transformations $L_{\beta\alpha} \in d\mathrm{SO}(5,1)$ rescale to simplified versions of the generators of $\mathcal{L}_{\mathcal{X}}(\mathbf{q}_0)$ in $\mathcal{L}_{\mathcal{X}}(\mathbf{q})$. The 15 variables $L_{\beta\alpha}$ require four quantum constants $\mathbf{h} = (\delta x, \delta p, \delta L, \delta r)$, but $\delta L = \hbar = 1$ for Lorentz invariance:

$$\widehat{L}_{\nu\mu} = L_{\nu\mu}, \quad \widehat{x}_\mu = \delta x\, L_{\mu \mathrm{X}}, \quad \widehat{p}_\mu = \delta p\, L_{\mu \mathrm{Y}}, \quad \widehat{r} = \delta r\, L_{\mathrm{XY}}. \tag{12}$$

The maximum eigenvalue of $-(L_{\alpha\beta})^2$ is the same for any spatial $(\alpha\beta)$ plane, a new quantum number we write as $l^2_{\mathcal{X}}$. Evidently in the singular limit we must have

$$\delta x\, \delta p = l_{\mathcal{X}}\, \delta r\, \hbar \tag{13}$$

and we might as well impose this in general.

This simplification converts the compound Lie algebra $\mathcal{L}_{\mathcal{X}}(\mathbf{q}_0)$ to a simple Lie algebra $\mathcal{L}_{\mathcal{X}}(\mathbf{q})$ with generators $L_{\alpha\beta}$. The canonical commutation relations survive in the simplified form

$$[x^\mu(h), p_\nu] = \delta^\mu_\nu \frac{\delta x \delta p}{\delta r} r. \tag{14}$$

We construct a quantum space STiME $= \widehat{\mathcal{X}} = \mathcal{X}(\mathbf{q})$ from its Lie algebra $\mathcal{L}_{\mathcal{X}}$ by specifying an irreducible matrix represention $R(\mathbf{h})\mathcal{L}_{\mathcal{X}}$, whose algebra is then the operational algebra of $\widehat{\mathcal{X}}$. The singular space-time algebra is an infinite-dimensional irreducible unitary representation $R(\mathbf{h}_0)\mathcal{L}_{\mathcal{X}}(\mathbf{c}_0)$ supported by the function space $L^2(\mathcal{X}(\mathbf{c}_0))$. To fix on one regularized STiME we must fix on quantum constants and quantum numbers $\mathbf{q}$ of the Lie algebra $\mathcal{L}_{\mathcal{X}}$, defining a preferred basis $\mathbf{B}$ in one irreducible representation $R(\mathbf{h})\mathcal{L}_{\mathcal{X}}(\mathbf{c})$.

And the Lie algebra $\mathcal{L}_{\mathcal{X}}$ is specified in turn by signatures.

The quadratic mode-vector space supporting the defining representation of $\mathcal{L}_{\mathcal{X}}$ is a 6-dimensional space $V_{\mathcal{X}}$. We form a high-dimensional representing vector space $R(\mathbf{h})V_{\mathcal{X}}$ with collective quantum constant $\mathbf{h}$, to support the physical representation $R(\mathbf{h})\mathcal{L}_{\mathcal{X}}$. The singular space is spanned by polynomials in the coordinates, and limits thereof.

An irreducible representation $R(\mathbf{c})\mathrm{SO}(5,1)$ is defined by eigenvalues c_n of the Casimir invariants C_n, the coefficient of z^n in the invariant characteristic polynomial

$$C(z) = \det(L - z\mathbf{1}) = \sum C_n z^n \tag{15}$$

for $L \in dR(\mathbf{h})\mathrm{SO}(5,1)$. C_n vanishes for odd n because $L \sim -L$, leaving C_2, C_4, C_6. As usual iL_{XY} has eigenvalue spectrum of the form $-l, -l+1, dots, l-1, l$. The extreme value l is a regulant and $2l$ is an integer. In the singular limit $l \to \infty$.

Let $\mathsf{L} = (L^i_j)$ be the matrix whose elements are the infinitesimal generators of $dR(\mathbf{c})\mathrm{So}(5,1)$; a matrix of matrices. Then for each $n \in \mathbb{N}$, $\mathrm{Tr}\,\mathsf{L}^n$ is another convenient invariant, whose value in the chosen representation we designate by $\Lambda^{(n)}$. In particular,

$$\Lambda^{(2)} = -(L_{\mathrm{XY}})^2 - L^{\mathrm{X}\mu}L_{\mathrm{X}\mu} - L^{\mathrm{Y}\mu}L_{\mathrm{Y}\mu} + L^{\nu}{}_{\mu}L^{\mu}{}_{\nu} \tag{16}$$

The cross-terms $-L^{\mathrm{X}\mu}L_{\mathrm{X}\mu} - L^{\mathrm{Y}\mu}L_{\mathrm{Y}\mu}$ have vanishing expectation for any eigenvector of L_{XY} by the generalized uncertainty inequality. Then

$$\Lambda^{(2)} = l^2 - (\delta x)^{-2}\langle \widehat{x}^{\mu}\widehat{x}_{\mu}\rangle - (\delta p)^{-2}\langle \widehat{p}^{\mu}\widehat{p}_{\mu}\rangle + \langle L^{\nu}{}_{\mu}L^{\mu}{}_{\nu}\rangle \approx l^2 \tag{17}$$

holds for the vacuum, as an eigenvector of extreme L_{XY}. In the correspondence domain one may drop the circumflexes.

This is a simplified Klein-Gordon equation with a “mass” term that depends on the STiME cooordinates and angular momentum. Wigner taught us that the scalar fields supporting irreducible representations of the Poincaré group obey Klein-Gordon wave equations. Naturally a simplified group leads to a simplified wave equation.

Similarly

$$\forall n \in \mathbb{N} \Big| \quad c^{(2n)} - l^{2n} \approx 0 = c^{(2n+1)} \tag{18}$$

are polynomial conditions on $\widehat{x}^{\mu}$, $\widehat{p}_{\mu}$, and $L^{\mu}{}_{\nu}$ with coefficients depending on $\mathbf{h}$ and l.

Raising the dimension of the group has increased the number of invariants and wave equations.

The algebra $\widehat{A} = \widehat{A}(\widehat{\mathcal{X}})$ of coordinate variables of the regular STiME quantum space $\widehat{\mathcal{X}} := \mathcal{X}(\mathbf{h})$ is the operator algebra of the vector space $R(\mathbf{c})V_{\mathcal{X}}$ that we have just constructed:

$$\widehat{A} := \mathrm{Endo}\ R(\mathbf{c})V_{\mathcal{X}}. \tag{19}$$

Each factor in $R(\mathbf{c})V_{\mathcal{X}}$ contributes angular momentum ± 1 or 0 to each generator $L_{\alpha\beta}$ of $\mathcal{L}_{\mathcal{X}}(h)$, so the eigenvalue of $iR(\mathbf{c})L_{\alpha\beta}$ varies from $-l$ to l in steps of 1. Now the space-time coordinates and the energy-momenta are unified under the Lie group generated by $R(\mathbf{c})\mathcal{L}_{\mathcal{X}}$. Each has a discrete bounded spectrum with $2L+1$ values

$x = i\delta x\, m,\ p = i\delta p\, m$, for $m \in \mathbb{Z}, |m| \in l+1$. Both operators are elements of the STiME operator algebra $A\mathcal{X}_h := \text{Endo}\ R(\mathbf{c})\mathcal{L}_{\mathcal{X}}$, which replaces $\mathrm{L}^2(\mathcal{X}_0)$.

The regular quantum point of STiME can be represented as a series of l more elementary processes or chronons, all identical, a bosonic ensemble constrained to a fixed number l of elements. The chronon is a minkowskion in this model.

Next we set up a singular scalar q/c field theory on the singular quantum space-time so that we can regularize it in §6.3.

6.2 Singular field Lie algebra

We label the singular q/c limit with a suffix $(\mathbf{q}_0)$ and generic q/q case with $(\mathbf{q})$, dropping the circumflex, where $\mathbf{q}$ is a collection of quantum constants and quantum numbers to be specified.

In the c scalar theory a history f of the field is a pair $(f(\cdot), p_f(\cdot)$ of a field function $f : \mathcal{X} \to \mathbb{R}$ on space-time, and a contragredient momentum function $p_f : \mathcal{X} \to \mathbb{R}$. The space of such c histories is, aside from continuity requirements,

$$\mathcal{F} = \mathbb{R}^{\mathcal{X}} =: \mathrm{D}\mathcal{X}, \tag{20}$$

a kind of linear dual of $\mathcal{X}$. The c functional Lie algebra is commutative:

$$\begin{aligned}
f(x)f(x') - f(x')f(x) &= 0,\\
p_f(x)p_f(x') - p_f(x')p_f(x) &= 0,\\
p_f(x)f(x') - f(x')p_f(x) &= 0,\\
f^\dagger + f &= 0,\\
{p_f}^\dagger + p_f &= 0.
\end{aligned} \tag{21}$$

The bosonic aggregate, or the quantum field, has the formal functional Lie algebra $L_{\mathcal{F}}(\mathbf{q}_0)$ generated by the operators on $\mathrm{D}\mathcal{X}$ of multiplication by $f(\cdot)$ variational differentiation $p_f = \delta_f(x) := \delta/\delta f(x)$, and the central i, subject now to the canonical relations of a $d\mathrm{H}(\infty)$,

$$\begin{aligned}
f(x)f(x') - f(x')f(x) &= 0\\
p_f(x)p_f(x') - p_f(x')p_f(x) &= 0\\
p_f(x)f(x') - f(x')p_f(x) + i\hbar\delta(x-x') &= 0\\
f^\dagger + f &= 0\\
{p_f}^\dagger + p_f &= 0.
\end{aligned} \tag{22}$$

for all $x, x' \in \mathcal{X}$. We make both f and p_f anti-Hermitian with incorporated factors of i where necessary, for the sake of the development to come.

Because of the two-story construction the operators x^μ and ∂_μ in the space-time Lie algebra $d\mathcal{L}_{\mathcal{X}}$ can also act on the field functional Lie algebra $d\mathcal{L}_{\mathcal{F}}$, with obvious commutation relations.

The element $i\hbar$ is a complete set of central invariants of this functional Lie algebra. The canonically quantized scalar field is a bosonic aggregate of individuals whose mode-vector space is $\mathrm{L}^2(\mathcal{M})$.

This is the theory we simplify next.

6.3 Regular field Lie algebra

The q/c scalar field is a bosonic aggregate. The Lie algebra of bosonic statistics is unstable, compound. We simplify it now to a simple, stable, and finite near-bosonic statistics.

We use the Lie-algebraic procedure $\Sigma_{\mathcal{L}}$ of (7). A fixed io mode-vector space V for an individual quantum I is given, and we give a Lie algebra $\mathcal{L}$ on V, with a structure tensor X close to the bosonic. It is convenient to give X on an ι-labeled replica of V in case there are other Lie algebra structures already defined on V, as in multiple quantification.

Then the mode-vector space ΣV of the quantified system is determined by the Lie algebra $\mathcal{L}$ and quantum constants $\mathbf{c}$. A vacuum projection $P(\text{vac}) \in \Sigma V$ then determines the vector space $\Sigma V \ P(\text{vac})$ as a mode-vector space for the quantified system.

In the case of the singular q/c boson quantification $\mathcal{L}$ is the functional Lie algebra $\text{Canon}_{+}\iota^{\dagger}V$, defined by the bosonic commutation relations on the union $V = V_{\text{I}} \cup V_{\text{O}}$ of the input and output mode-vectors of the system. V is a partial vector space; addition works within each term but not between them.

Then the bosonic operational algebra of the aggregate of individuals I is the target algebra $A(\mathbf{c})$ of the irreducible unitary representation $R(\mathbf{c})$ of $\mathcal{L}$ with the central invariant $c = i\hbar$ specified.

The usual creator and annihilator of the many-quantum (or quantified) theory associated with the mode-vectors v and $v^{\dagger}$ of the one-quantum theory are left multiplication $\iota^{\dagger}v$ and differentiation $(\iota^{\dagger}v)^{\dagger} = v^{\dagger}\iota$ with respect to $\iota^{\dagger}v$.

If the $v_n \in V$ form a basis of input vectors with dual output vectors, the corresponding creation/annihilation operators $a_n := \iota^{\dagger}v_n$, $c^n := v^{n\dagger}\iota$ obey

$$\begin{aligned} c^n a_m - a_m c^n - i\hbar\delta^n_m &= 0, \\ c^n c^m - c^m c^n &= 0, \\ a_n a_m - a_m a_n &= 0. \end{aligned} \tag{23}$$

This algebra is doubly infinite-dimensional: once because bosonic quantification turns each dimension on the one-quantum space into an infinity of dimensions in the many-quantum space, and once because the one-quantum space has an infinity of dimensions, because space-time is infinite and continuous. The space-time infinity is again bosonic, arising from the fact that space-time is a bosonic aggregate of minkowskions,

A boson Lie algebra on a $2N$ dimensional io vector space is a contraction of an SO Lie algebra on $2N+2$ dimensions. To simplify the relations (23) we first transform bosonic variables a_n, c^n to canonical anti-Hermitian variables $q_n = -{q_n}^{\dagger}$, $p_n = -{p_n}^{\dagger}$ $(n \in \mathbb{N})$ using the imaginary unit i:

$$a_n = \frac{q_n/\delta q + ip_n/\delta p}{\sqrt{2}}, \quad c_n = \frac{q_n/\delta q - ip_n/\delta p}{\sqrt{2}}, \tag{24}$$

including quantum constants $\delta q, \delta p, \delta r$ for dimensional reasons. Then we introduce two extra real dimensions with indices X', Y' forming a real vector space $V \oplus \mathbf{2}$ with vector indices $\alpha, \beta = 0, \ldots, N-1, ,\text{Y}$. A symmetric metric $\dagger : (V \oplus \mathbf{2}) \to (V \oplus \mathbf{2})^{\text{D}}$ defines an orthogonal Lie algebra $d\text{SO}(V \oplus \mathbf{2})$ generated by $(N+2) \times (N+2)$ matrices $L_{\beta\alpha}$, anti-Hermitian with respect to $\dagger$. We represent the simplified simple-bosonic creators and annihilators in the simple Lie algebra $d\mathrm{SO}(V \oplus \mathbf{2})$:

$$\widehat{q}^n := \delta q L^n{}_{\mathrm{X}'}, \quad \widehat{p}_n = \delta p L^{\mathrm{Y}}{}_n, \quad \widehat{\imath} := \delta r L^{\mathrm{Y}}{}_{\mathrm{X}'}, \tag{25}$$

For an alternative representation see Baugh [4].

The space-time simplification introduced a large quantum number $l_{\mathcal{X}}$, setting the maximum of the space-time iL_{XY}, and approaching ∞ in the singular limit. This determines the dimension N of the space-time mode-vector space. Now the field algebra simplification introduces another large quantum number $l_{\mathcal{F}}$ determining the maximum eigenvalue of the field $L_{\mathrm{X}'\mathrm{Y}'}$. The simplified relations include

$$[\widehat{q}^m, \widehat{p}_n] = i\delta q \delta p \left(\delta^n_m L^{\mathrm{X}'}_{\mathrm{Y}'}\right) \to i\hbar \delta^n_m \tag{26}$$

We infer that

$$l \, \delta q \, \delta p = \hbar \tag{27}$$

This simplifies the Lie algebra. Now we must simplify its representation. To construct the physical variables, which typically have many more eigenvalues, we must pass from the given low-dimensional Lie algebra to a suitable irreducible orthogonal representation of dimension large enough to pass for infinite.

In the singular theory this representation is the bosonic quantification of the underlying Lie algebra, unique up to one quantum constant $\hbar$ but infinite-dimensional. Here in the regular theory the space-time Lie algebra is simple, that of $\o SO(5,1)$, and its representation has finite dimension $\widehat{D}_{\mathcal{X}}$, defined by central invariants $\widehat{\mathbf{c}}_{\mathcal{X}}$. The representation of the physical basis is defined by quantum constants $\widehat{\mathbf{h}}_{\mathcal{X}}$ close to their singular values. Then the simplified boson Lie algebra $\widehat{\mathcal{L}}_{\mathcal{F}}$ has finite dimension

$$\widehat{D}_{\mathcal{F}} = 2\overline{D}_{\mathcal{X}} + 2. \tag{28}$$

The operational algebra has finite dimension determined by the central invariants $\widehat{c}_{\mathcal{F}}$ of the simplified field Lie algebra.

6.4 Singular scalar dynamics

The usual scalar Green's function is

$$G(x'_1, \ldots, x'_n) = \langle \mathrm{vac}|\phi(x'_1), \ldots, \phi(x'_n))|\mathrm{vac}\rangle \tag{29}$$

Here x'_1 is a collection of c numbers, eigenvalues of the coordinate operators $x = (x^\mu)$, and $\phi(x'_1)$ is a creation/annihilation operator associated with the position eigenvalue x'_1.

The construct G is covariant under the unitary group of basis changes for the space F of fields $\phi(x')$. Any orthonormal frame $\{\phi_\alpha\}$ for the mode-vector space of a single boson defines a generalized Green's function

$$G_{\alpha_1,\ldots,\alpha_n} = \langle \mathrm{vac}|\phi_{\alpha_1}, \ldots, \phi_{\alpha_n}|\mathrm{vac}\rangle \tag{30}$$

This form can survive the simplifying that we carry out. The nature of the one-quantum mode-vector, however, changes discontinuously at the singular limit. For example, in c space-time the coordinates x^μ all commute, and so their eigenvalues can label the mode-vector $\phi_{x'}$. But in the simplified quantum space (STiME), space-time coordinates $\widehat{x}^\mu$ do not commute and their eigenvalues cannot label a basis. Instead there are commuting variables $t = \delta t\, L_{0\mathrm{X}}$, $p_x = \delta p\, L_{1\mathrm{Y}}$, and L_{23}, which may be supplemented by the quantum numbers c_2, c_4, c_6 as necessary to make a complete commuting set. To recover the singular Green's function from the regular we must construct coherent states that are only approximately eigenvectors of all the $\widehat{x}^\mu$.

The vacuum mode-vector $|\mathrm{vac}\rangle$ of the singular quantum theory is defined by its amplitude, which has the Lagrangian form

$$\langle\phi(\cdot)|\mathrm{vac}\rangle := N \exp i \left[\int d^4x\, L(\phi(x), \partial_\mu\phi(x))\right] =: N \exp iA. \tag{31}$$

in which $A = A[\phi(.)]$ is the action integral of the exponent. This gives an amplitude for each field history $\phi(\cdot)$.

The singular dynamical theory we simplify is that of a free scalar meson, with Lagrangian density

$$L(\phi(x), \partial_\mu\phi(x)) := -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi(x)^2 \tag{32}$$

6.5 Regular scalar dynamics

The free field or many-quantum action A is constructed from the one-quantum anti-symmetric operator

$$A_1 := ip^\mu p_\mu + im^2 \tag{33}$$

by quantification. To quantify A_1, we first make explicit the mode-vectors ϕ_x and their duals $\phi_x{}^\dagger$ that enter into it, and then give a Lie-algebra on them. We choose an x basis only for its familiarity:

$$A_1 = N \int d^4x\, \phi_x L^{xx'} \phi_{x'}{}^\dagger, \tag{34}$$

with a singular normalizer N and a singular kernel $L^{xx'}$. Then quantification replaces the one-quantum mode-vectors ϕ_x and their duals $\phi_x{}^\dagger$ by many-body operators $\iota\phi_x$ and $\phi_x{}^\dagger\iota^\dagger$ obeying bosonic commutation relations, defining the same singular algebra as a particle in infinite-dimensional space. The result is the singular action A of (31), now written

$$A = N \int d^4x\, \iota\phi_x L^{xx'} \phi_x{}^\dagger\iota^\dagger = \iota A_1 \iota^\dagger \tag{35}$$

To simplify A we need only simplify A_1.

To be sure, the algebra of ι and $\iota^\dagger$ is singular and infinite dimensional. Perhaps it too can be regularized. This would modify the q set theory to allow membership loops, as in Finsler set theory. But we do not iterate ι, so it introduces no singularities, and we leave ι fixed.

To simplify the action A_1 we simplify each operator in it. As usual, quantization requires us to order operators that no longer commute so that their product remains antisymmetric. For economy we choose the order

$$\widehat{A}_1 = \widehat{p}^\mu\widehat{\imath}\widehat{p}_\mu + m^2\widehat{\imath}. \tag{36}$$

The simplified action is then

$$\widehat{A} = \widehat{\iota}\widehat{A}_1\widehat{\iota}^\dagger \tag{37}$$

The simplified creators and annihilators obey simplified bosonic commutation relations, those of $d\mathrm{SO}(M)$ with cosmologically large M.

Obviously, this is finite and so is the normalization constant $\widehat{N}$ replacing the infinite constant N. The exact Lorentz invariance and the approximate medium-energy Poincaré invariance are also plausible.

This simplified action seriously breaks the simplified symmetry group, and more symmetric ones that are still good approximations to the singular actgion in the correspondence domain are readily available. They go beyond the scope of this paper.

7 Results

We have used general quantization to convert the usual singular theory of the scalar meson to a finite theory with nearly the same algebras and symmetries in a correspondence domain. This toy taught us how to general-quantize Minkowski space-time and bosonic statistics, and how to supply a relativistic finite dynamics to go with the finite quantum kinematics.

Simplifying space-time quantizes momentum-energy and space-time. It produces a finite unified quantum space-time-i-momentum-energy space STiME.

The quantization of Minkowski space-time exhibited here has chronons with simplified bosonic statistics and the symmetry group $\mathrm{SO}(5,1)$. It is a transient theory but some of its features are typical. For one thing, it is intrinsically non-local in both space and momentum variables with respective non-localities δx and δp. It also has an invariant integer parameter N, a maximum number of elementary processes. The ether crystallization breaks Born reciprocity in the singular limit $\delta x \to 0$, $\delta p \to \infty$, $\mathsf{N} \to \infty$, and makes the singular limit theory local in space-time but not in energy-momentum. That is, in a single interaction there is no finite change in position or time, but an arbitrarily large change in momentum and energy; the standard assumption.

The principle difference between this approach and most others is that we take seriously the partition of the theory into logical levels, each with its algebra, and preserve these algebras, with small changes, throughout the construction. This contrasts, for example, with approaches to quantum field theory that discretize space-time, discarding the Lorentz invariance, and then take a limit. Under general quantization the system determines its own quanta and requires no ad hoc discretization.

The correspondence principle fixes some combinations of the new quantum constants, quantum numbers, and regulants, leaving the rest to experiment. No infinite renormalization is needed.

Several discrete choices have to be left to experiment. For example the simplicity principle is equally satisfied along the real, complex, and quaternionic lines of simple Lie algebras. We chose the real line mainly because it is easiest and in some sense simplest, but nature may not take the way that is easiest or simplest for us.

We give necessary conditions on the defining parameters for the finite theory to converge to the usual theory in some appropriately weak sense, but we have not shown they are sufficient. This question may be sensitive to the theory under study. We have not proven that these finite results agree well enough with the finite results of the usual singular theory where they should but it seems plausible. Approximating the regular discrete spectrum by a singular continuous one is a somewhat delicate non-uniform convergence even for the harmonic oscillator.

Such a change in the most basic algebraic relation of quantum theory and in the Heisenberg Uncertainty Relation has many experimental consequences to be developed. Vilela Mendes points out that it permits a serious reduction in phase space $\Delta p\,\Delta q$ at high energy that may explain the GZK anomaly [40].

We suspend our study of the scalar field for now in order to general-quantize more basic systems, the gauge fields mediating physical interactions.

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