

# Fundamental Concepts of Elementary Mathematics – Math 107B

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# 1 Fractions

## 1.1 Definition of Rational Numbers

As a child you first learned to count  $1, 2, 3, 4, \dots$ . In mathematics we call this set of numbers the *Natural Numbers*, we will use  $\mathbb{N}$  to represent the set of natural numbers. In other words,  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ . If we add the number zero to the set of natural numbers, we get the *Whole Numbers* which we denote by  $\mathbb{W}$ . In other words,  $\mathbb{W} = \{0, 1, 2, 3, \dots\}$ .

If we add two numbers in the set of whole numbers, the result will be in the set of whole numbers. We say that the set of whole numbers is *closed* under addition. However, if we subtract any two whole numbers, the result may not be in the set of whole numbers. For example,  $3 - 5$  is not in the set of whole numbers. So we say that the whole numbers are not closed under subtraction.

Since our answer doesn't lie in the set of whole numbers, we must extend the set to include all possible answers to a subtraction problem. In other words, we need negative numbers. We call this new set of numbers (which include whole numbers) the set of *Integers* and we use  $\mathbb{Z}$  to denote the set of integers. In other words,  $\mathbb{Z} = \{\dots, -2, -1, 1, 2, \dots\}$ . Notice that the set of integers is closed under both addition and subtraction. In fact, it is also closed under multiplication as well. Unfortunately, the set of integers is not closed under division, so we would like to again extend the set of numbers we can work with.

**Definition 1.1.** A *Rational Number* is any number that can be written in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$ . We denote the set of *Rational Numbers* by  $\mathbb{Q}$ . Thus in set notation we have:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

A quick comment on vocabulary. For a rational number  $\frac{a}{b}$ ,  $a$  is called the *numerator* and  $b$  is called the *denominator*.

### Example 1.2.

1. Is  $\frac{3}{5}$  a rational number? Yes, because 3 and 5 are integers and  $5 \neq 0$ .
2. Is  $\frac{2}{9} \in \mathbb{Q}$ ? Yes, because 2 and 9 are integers and  $9 \neq 0$ .
3. Is  $\frac{3}{0}$  a rational number? No, because the bottom is zero.

4. Is  $-\frac{1}{4}$  a rational number? Yes, because  $-1$  and  $4$  are integers and  $4 \neq 0$ .
5. Is  $-\frac{1}{4}$  a rational number? (Notice the difference between this question and the previous question.) We do not know enough yet to answer this question because  $-\frac{1}{4}$  is not of the form  $\frac{a}{b}$ , but we will eventually find out the answer is yes.

## 1.2 Parts of the Whole Model

We now know what a rational number looks like, namely it can be written in the form  $\frac{a}{b}$ . Unfortunately the definition of a rational number does not give us any sort of indication what this  $\frac{a}{b}$  thing actually is. Thus our first task is to give meaning to something like  $\frac{3}{5}$ .

### Definition 1.3. *Parts of the Whole*

Let  $a$  be a whole number and  $b$  be a natural number.

1. Designate a whole.
2. Break the whole into  $b$  equal pieces.
3.  $\frac{a}{b}$  is represented by  $a$  pieces.

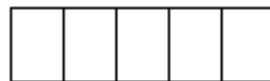
According to this definition, any whole can be used. For example it can be a candy bar (which I like to use), or a pie, or a pizza, or a classroom, or whatever you'd like.

### Example 1.4. Use a parts of the whole model to represent $\frac{3}{5}$ .

We begin by designating a whole. I am going to use a candy bar, which I will draw as a rectangle.



Now we need to break the whole into 5 equal pieces.



Now we shade 3 of these pieces.



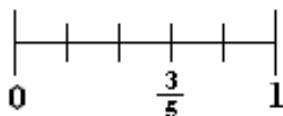
So the fraction  $\frac{3}{5}$  is represented by the shaded region.

The key to the above example was to designate a whole. Once that was done, everything followed easily. The idea here is that the whole is supposed to represent the number 1. In other words, we could represent the number one by one whole candy bar, or one whole pie, or one whole pizza, etc.

We have now put meaning to a fraction  $\frac{a}{b}$  where  $a$  is a nonnegative integer and  $b$  is a positive integer. The above example gives us a way to represent  $\frac{3}{5}$ , but is  $\frac{3}{5}$  actually a number? If so, where is it on the number line? In order to answer these questions, we must be able to use the parts of the whole model on the number line. As stated above, the only issue we have is what the whole is on the number line. If the whole is supposed to represent the number 1, then on the number line the whole must be the interval from 0 to 1.

**Example 1.5.** Locate  $\frac{3}{5}$  on the number line.

As discussed above, the whole is the interval from 0 to 1. We then need to break the interval into 5 equal pieces and 3 of these pieces represents  $\frac{3}{5}$ . Therefore we can use parts of the whole model to locate  $\frac{3}{5}$  on the number line.



We have now seen two examples of the parts of the whole model. The two wholes that I used in the previous examples are the two I tend to use the most, namely candy bars (i.e. rectangles) and the number line. There are plenty of other plausible choices for a whole of course, and we will discuss these occasionally.

Now that we have working knowledge of certain types of fractions, let's address a few questions.

**Question 1.6.** How does the parts of the whole model work if the numerator is larger than the denominator?

Let's consider a specific example here. Let's say  $\frac{5}{3}$ . I first need to designate a whole, then break it into three equal pieces. I now need to take five of those pieces. Of course the issue here is that I need five pieces to

represent  $\frac{5}{3}$ , but there are less than five pieces. I have designated a whole, but there is nothing to stop me from using more than one whole (as long as the wholes are all the same). Therefore I simply use another copy of the whole, break it into three pieces. Now between the two wholes I have the five pieces needed to represent  $\frac{5}{3}$ .

**Question 1.7.** In the definition of a rational number there is the condition that  $b$  cannot equal 0. Use the parts of the whole model to explain why  $b$  cannot equal 0.

We begin by designating a whole. Let's say a candy bar. We must then break the candy bar into 0 equal pieces. This is impossible, which is why we cannot have 0 in the bottom of our fraction.

**Question 1.8.** Does  $\frac{0}{4}$  equal 0?

Yes. Consider representing  $\frac{0}{4}$  on the number line. We would break the interval from 0 to 1 into 4 equal pieces and take 0 of them, which means I would be sitting at 0.

There is nothing special about 4 in the above example. In other words,  $\frac{0}{b}$  equals 0 for any natural number  $b$ . So we have found that 0 is a rational number because it can be represented as  $\frac{0}{b}$ . This is our first example (in this class) of a number that can be expressed in more than one way. Similarly 4 can be expressed as  $\frac{4}{1}$ . In fact, any whole number  $a$  can be expressed as  $\frac{a}{1}$ . Thus all whole numbers are rational numbers. In other words,  $\mathbb{W} \subseteq \mathbb{Q}$ .

From the above discussion, we now know that all whole numbers can be expressed as a fraction. However, the way we express them as a fraction is not unique. For example, we could have expressed 4 as  $\frac{8}{2}$ , which implies that  $\frac{4}{1} = \frac{8}{2}$ . The notion that two fractions can be equal even though they are not expressed in the same way can be difficult for students to grasp. The following example introduces this notion.

**Example 1.9.** Use a parts of the whole model to show  $\frac{2}{3}$  is equal to  $\frac{4}{6}$ .

We first represent  $\frac{2}{3}$  using parts of the whole model.



The shaded area represents  $\frac{2}{3}$ . We now break each of the three equal pieces into two equal pieces. These extra cuts are shown in red.



The whole is now broken into a total of six pieces, four of which are shaded. Therefore the shaded area now represents  $\frac{4}{6}$ . However, the shaded area did not change. Thus  $\frac{2}{3}$  is equal to  $\frac{4}{6}$ .

**Question 1.10.** Consider the parts of the whole model for  $\frac{2}{3}$ . How many pieces would you break each of the original pieces into, to show that  $\frac{2}{3}$  is equal to  $\frac{14}{21}$ ?

The parts of the whole representation of  $\frac{14}{21}$  must have 21 pieces in the whole. Therefore we need to turn three pieces into 21 pieces. We can do this by breaking each of the three pieces into 7 pieces (because  $3 \times 7 = 21$ ). Notice that the shaded area, which was originally two pieces, will now consist of 14 pieces (because  $2 \times 7 = 14$ ). Thus, breaking each of the three pieces into 7 pieces will show that  $\frac{2}{3}$  is equal to  $\frac{14}{21}$ .

**Question 1.11.** Consider the parts of the whole model for  $\frac{2}{3}$ . If we break each of the three pieces into five pieces. What fraction will be represented?

The whole originally consists of three pieces. After further breaking there will be  $3 \times 5$ , or 15, total pieces. Similarly, the shaded area was originally two pieces. After further breaking there will be  $2 \times 5$ , or 10, shaded pieces. Thus the final representation has 10 shaded pieces and 15 pieces in the whole, which represents the fraction  $\frac{10}{15}$ .

From the above discussion there are two things that should be noted. Firstly, there are infinitely many fractions that are equal to  $\frac{2}{3}$ . We can see this by cutting the pieces into 2, 3, 4, 5, ... etc. pieces. Secondly, we should be able to visualize the process of showing two fractions are equal without actually having to draw a picture. In other words, we should be able to describe algebraic properties that correspond to the pictures we drew.

Notice in the previous examples, that each time we broke pieces further, the number showed up as a multiplier to the numerator and the denominator of the original fraction. Thus if we take the fraction  $\frac{2}{3}$  and multiply the top and bottom of the fraction by 4 for example, we will get  $\frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}$ . Multiplying by four on the top and the bottom will just correspond to breaking each of the original pieces in the representation of  $\frac{2}{3}$  further into four pieces. Thus  $\frac{2}{3} = \frac{8}{12}$ . Because we would like to use the fact that we can multiply the top and bottom of a fraction to produce an equivalent fraction, we should give this property a name.

**Proposition 1.12. Fundamental Law of Fractions**

*Let  $n$  be a natural number, then  $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ .*

With this algebraic property in hand, we now want to get to the point where we can show two fractions are equal without having to draw pictures. We still want to be sure that we are only using properties, definitions, etc that we have learned (either in this class or 107A), therefore we should justify each equality. In other words, each time we say that two things are equal, we should be able to give a reason.

**Example 1.13.** Show algebraically that  $\frac{2}{3}$  is equal to  $\frac{4}{6}$ .

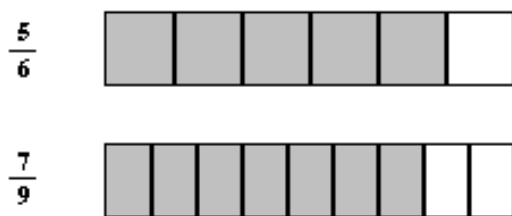
$$\frac{2}{3} \stackrel{(i)}{=} \frac{2 \cdot 2}{3 \cdot 2} \stackrel{(ii)}{=} \frac{4}{6}$$

The first equality, (i), is due to the Fundamental Law. The second equality, (ii), is due to the fact that we know how to multiply integers.

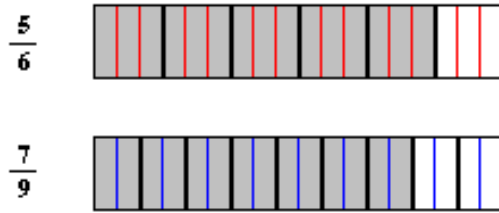
Now that we are comfortable showing two fractions are equivalent (both visually and algebraically), we can begin to compare fractions. When using parts of the whole model to compare fractions, we can sometimes “eye” the two pictures to determine which fraction is larger. This will not always work however, so we need to develop a method that does not rely on the perfection of our drawing. In particular, our goal is to get all the pieces the same size so that we can just count up the number of pieces associated to each fraction. Let’s take a look at an example.

**Example 1.14.** Use a parts of the whole model to compare the fractions  $\frac{5}{6}$  and  $\frac{7}{9}$ .

We begin by representing  $\frac{5}{6}$  and  $\frac{7}{9}$  using the same size whole.



To compare these two fractions, I need to get the same size pieces in each model. I will break each piece in the  $\frac{5}{6}$  model into 3 pieces (indicated by red below). I will break each piece in the  $\frac{7}{9}$  model into 2 pieces (indicated in blue below).



All the pieces are now the same size.  $\frac{5}{6}$  is represented by 15 pieces and  $\frac{7}{9}$  is represented by 14 pieces. Thus  $\frac{5}{6}$  is greater than  $\frac{7}{9}$ .

As with creating equivalent fractions, we should be able to write down the algebraic process involved in comparing two fractions. The key to remember here is that two fractions with the same denominator can be compared easily by just looking at their numerators. In the parts of the whole model, we wanted to get the same size pieces and this corresponds to getting a common denominator in the algebraic method. Let's look at the following example.

**Example 1.15.** Compare the fractions  $\frac{5}{6}$  and  $\frac{7}{9}$  algebraically. Justify all equalities.

$$\frac{5}{6} \stackrel{(i)}{=} \frac{5 \cdot 3}{6 \cdot 3} \stackrel{(ii)}{=} \frac{15}{18}$$

$$\frac{7}{9} \stackrel{(i)}{=} \frac{7 \cdot 2}{9 \cdot 2} \stackrel{(ii)}{=} \frac{14}{18}$$

We have expressed each of the fractions with denominator 18, thus we need only compare numerators. Since  $15 > 14$ , we can conclude that  $\frac{5}{6} > \frac{7}{9}$ .

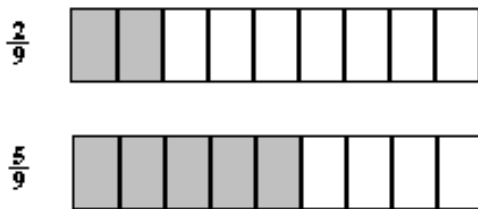
Justifications for equalities:

- (i) Fundamental Law
- (ii) Multiplication of integers

### 1.3 Addition and Subtraction

Now that we are comfortable with fractions both algebraically and as parts of a whole, we can begin adding fractions. As in the previous section we will begin by looking at parts of the whole and then translate our findings to algebraic definitions and properties.

**Example 1.16.** Use a parts of the whole model to find the sum  $\frac{2}{9} + \frac{5}{9}$ .



All the pieces are the same size. There are 9 pieces in a whole and all together there are 7 shaded pieces. Thus  $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$ .

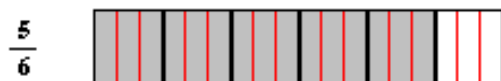
The above example was quite simple because all the pieces were the same size. If we are given a problem where the denominators are not the same, we then need to make all the pieces the same size so that we can combine them. When comparing fractions in the previous section we had the same issue, therefore we will proceed in the same way. The only difference is that rather than comparing the number of shaded pieces in the end we will instead combine them. Let's consider the following example.

**Example 1.17.** Use a parts of the whole model to find the sum  $\frac{5}{6} + \frac{7}{9}$ .

We begin by representing  $\frac{5}{6}$  and  $\frac{7}{9}$  using the same size whole.



To add these two fractions, I need to get the same size pieces in each model. I will break each piece in the  $\frac{5}{6}$  model into 3 pieces (indicated by red below). I will break each piece in the  $\frac{7}{9}$  model into 2 pieces (indicated in blue below).



All the pieces are now the same size.  $\frac{5}{6}$  is represented by 15 pieces and  $\frac{7}{9}$  is represented by 14 pieces. All together we have 29 shaded pieces. We also know there are 18 pieces in a whole. Thus  $\frac{5}{6} + \frac{7}{9} = \frac{29}{18}$ .

We now need to make the transition from parts of the whole models to algebraically computing the sum. From example 1.16 it is clear how to add fractions with a common denominator. Since all the pieces are the same size we are able to just count them up. In other words, the seven in example 1.16 came from the 2 shaded pieces from the first model plus the 5 shaded

pieces from the second model. Therefore we have the following definition of addition of fractions.

**Definition 1.18.** *Addition of Fractions*

Let  $\frac{a}{c}$  and  $\frac{b}{c}$  be rational numbers, then  $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ .

Unfortunately, the above definition only deals with summands that have a common denominator. We therefore need to make the transition from example 1.17 to computing the sum algebraically. The initial steps are the same as in the previous section. Let's consider the following example.

**Example 1.19.** Find the sum  $\frac{5}{6} + \frac{7}{9}$  algebraically. Justify each equality.

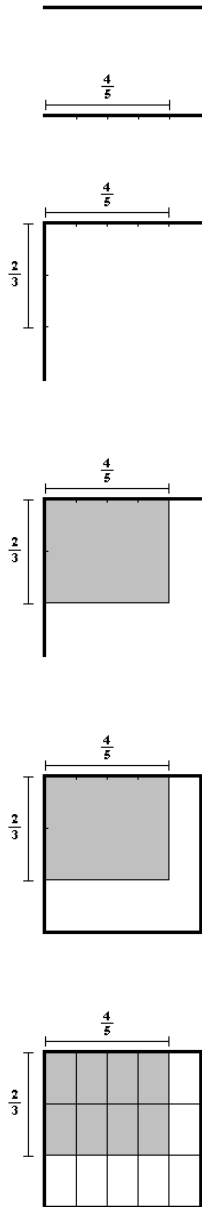
$$\frac{5}{6} + \frac{7}{9} \stackrel{(i)}{=} \frac{5 \cdot 3}{6 \cdot 3} + \frac{7 \cdot 2}{9 \cdot 2} \stackrel{(ii)}{=} \frac{15}{18} + \frac{14}{18} \stackrel{(iii)}{=} \frac{29}{18}$$

- (i) Fundamental Law
- (ii) Multiplication in  $\mathbb{Z}$
- (iii) Addition of Fractions

## 1.4 Multiplication

While teaching students to add fractions we really push the fact that we have to get a common denominator. Now when we teach them to multiply fractions we are going to tell them they don't need a common denominator. Therefore we would like the students to be able to see why that is. We are going to rely on the students knowledge of area. They understand how to find the area of a rectangle. For example, suppose you have a rectangle that is 5 inches by 4 inches. Then the area of the rectangle is 20 in<sup>2</sup>. The students understand two things here: 1) To find the area of a rectangle you multiply the two dimensions and 2) the area being 20 in<sup>2</sup> means 20 1 inch by 1 inch squares will fill up the rectangle. We can use their knowledge of area to teach them to multiply two fractions. We can turn the multiplication problem into a parts of the whole question, but the key to remember here is that the whole is a 1 by 1 square. Let's consider the following example.

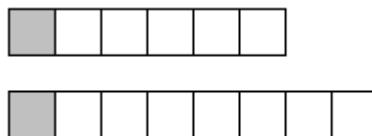
**Example 1.20.** Compute the product  $\frac{2}{3} \cdot \frac{4}{5}$ .



## 1.5 Division

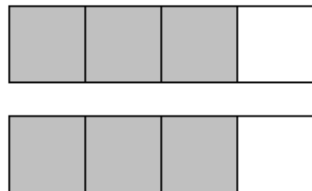
## 1.6 Problems

1. Explain why the parts of the whole model cannot be used to understand the following fractions. Be sure to indicate specifically where the problem occurs in the parts of the whole model.
  - (a)  $\frac{-2}{3}$
  - (b)  $\frac{2}{-3}$
2.
  - (a) Use a “parts of the whole” model to determine which fraction is larger,  $\frac{5}{6}$  or  $\frac{7}{9}$ . Be sure to explain all steps carefully.
  - (b) Determine algebraically which fraction is larger,  $\frac{5}{6}$  or  $\frac{7}{9}$ . Justify each equality.
  - (c) Where does the multiplier in the Fundamental Law of Fractions show up in your parts of the whole model above?
3. Four friends want to share six candy bars fairly. Considering we do not understand what  $6 \div 4$  means yet, use drawings and explain how to determine how much each friend gets.
4. Max is comparing the fractions  $\frac{1}{6}$  and  $\frac{1}{8}$ . He draws the following pictures and makes the conclusion that the two fractions are equal.

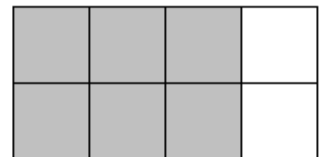


- (a) According to his pictures and his conclusion, what does Max understand about fractions?
  - (b) According to his pictures and his conclusion, what does Max not understand about fractions?
  - (c) What would you say to Max to help him.
5.
    - (a) Max claims that if we have two positive rational numbers, the one with the greatest numerator is the greatest. Is Max correct? If so, explain why. If not, explain why not and give a counterexample.

- (b) Mary claims that if we have two positive rational numbers, the one with the greatest denominator is the least. Is Mary correct? If so, explain why. If not, explain why not and give a counterexample.
6. Consider the set of fractions equivalent to  $\frac{1}{3}$ . If you plot points on a coordinate system where the numerator is the  $x$ -coordinate and the denominator is the  $y$ -coordinate, what will the graph look like?
7. (a) Find 4 rational numbers between  $\frac{7}{9}$  and  $\frac{8}{9}$ . Be sure to show all work clearly.
- (b) Find 4 rational numbers between  $\frac{a}{c}$  and  $\frac{b}{c}$ . Be sure to show all work clearly.
- (c) Find 4 rational numbers between  $\frac{5}{6}$  and  $\frac{7}{8}$ . Be sure to show all work clearly.
- (d) Find  $n$  rational numbers between  $\frac{5}{6}$  and  $\frac{7}{8}$ . Be sure to show all work clearly.
8. In order to add two fractions why do the two fractions need to have a common denominator?
9. (a) Use a parts of the whole model to find the sum  $\frac{3}{4} + \frac{1}{6}$ .
- (b) Use algebraic properties to find the sum  $\frac{3}{4} + \frac{1}{6}$ . Be sure to justify each equality.
- (c) Describe where the steps in the algebraic method show up in your parts of the whole model.
10. Max says that  $\frac{3}{4} + \frac{3}{4} = \frac{6}{8}$  and uses this drawing to support his claim:



**Add them together**



- (a) What does Max understand well about fractions?
- (b) What misconception(s) is(are) leading Max to believe that  $\frac{3}{4} + \frac{3}{4}$  can equal  $\frac{6}{8}$ .
- (c) How would you respond to Max?

11. State and solve a word problem whose solution is the sum  $\frac{2}{3} + \frac{4}{5}$ .

12. State and solve a word problem whose solution is the difference  $\frac{4}{5} - \frac{2}{3}$ .

13. You give Max the problem  $2\frac{3}{4} + 1\frac{1}{5}$ . Max does the problem in the following way.

$$2 + 1 = 3$$

$$\frac{3}{4} + \frac{1}{5} = \frac{15}{20} + \frac{4}{20} = \frac{19}{20}$$

Therefore Max concludes that  $2\frac{3}{4} + 1\frac{1}{5} = 3\frac{19}{20}$ .

(a) Is Max's method and answer correct? If yes, why does it work? If not, explain why not.

(b) You then give Max the problem  $2\frac{3}{4} + 1\frac{2}{5}$ . He tries to do it the same way, but gets confused. What could be confusing Max? What can you say to help him?

14. You now give Max the problem  $-2\frac{3}{4} + 1\frac{1}{5}$ . Max does the problem in the following way.

$$-2 + 1 = -1$$

$$\frac{3}{4} + \frac{1}{5} = \frac{15}{20} + \frac{4}{20} = \frac{19}{20}$$

Therefore Max concludes that  $-2\frac{3}{4} + 1\frac{1}{5} = -1\frac{19}{20}$ .

Is Max's method and answer correct? If yes, why does it work? If not, explain why not and give the correct answer.

15. (a) Give the definition of  $-\frac{2}{3}$ .

- (b) Prove that  $\frac{2}{-3} = -\frac{2}{3}$ . (Be sure to justify each equality.)
- (c) Prove that  $\frac{a}{-b} = -\frac{a}{b}$ . (Be sure to justify each equality.)
16. Two thirds of the people in a conference room are men. Nobody leaves the room, but 10 more men and 10 more women enter the room.
- (a) After the men and women come in are there more men or more women in the room. Explain how you know.
- (b) After the men and women come in is it still true that two thirds of the people in the room are men? If so, explain why. If not, give a counterexample.
17. Use an area model to illustrate each of the following products of rational numbers.

(a)

$$\frac{1}{3} \cdot \frac{1}{4}$$

(c)

$$\frac{8}{5} \cdot \frac{2}{3}$$

(e)

$$2\frac{1}{3} \cdot \frac{3}{4}$$

(b)

$$\frac{2}{5} \cdot \frac{1}{4}$$

(d)

$$\frac{8}{5} \cdot \frac{3}{2}$$

(f)

$$2\frac{1}{3} \cdot 1\frac{1}{2}$$

18. State and solve a word problem whose solution is the product  $\frac{2}{5} \cdot \frac{1}{4}$ .
19. State and solve a word problem whose solution is the product  $2\frac{1}{3} \cdot \frac{3}{4}$ .
20. State and solve a word problem whose solution is the product  $2\frac{1}{3} \cdot 1\frac{1}{2}$ .
21. Suppose  $a$  and  $b$  are non-zero rational numbers, and suppose  $m$  and  $n$  are integers. Which of the following statements are true and which are false. Justify your answers.

(a)

$$a^m \cdot b^n = (ab)^{m+n}$$

(b)

$$a^m \cdot b^n = (ab)^{mn}$$

(c)

$$(a + b)^m = a^m + b^m$$

(d)

$$a^{mn} = a^m \cdot a^n$$

22. Suppose  $\frac{a}{b}$  is a positive rational number.

- (a) Choose a positive number that is smaller than 1. If you square your number is the result bigger or smaller than your original number?
- (b) Choose a positive number that is larger than 10 and not an integer. If you square your number is the result bigger or smaller than your original number?
- (c) State a conjecture that indicates which is bigger  $\frac{a}{b}$  or  $\left(\frac{a}{b}\right)^2$ .  
(Note: you may need to break this down into different cases depending on what  $\frac{a}{b}$  is.)
- (d) Prove your conjecture.

23. Recall that we defined negative exponents in class. We said that  $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ .

- (a) Using this definition, explain to Max how to compute  $2^{-3}$ .
- (b) If  $a$  is an integer, prove that  $a^{-n} = \frac{1}{a^n}$ .
- (c) Max says that since  $a^{-n} = \frac{1}{a^n}$ , then  $(a + b)^{-n} = \frac{1}{a^n} + \frac{1}{b^n}$ . Is Max correct? If yes, prove it. If not, give Max a counterexample and show him what  $(a + b)^{-n}$  does equal.

24. State and solve a word problem whose solution is the quotient  $2\frac{1}{3} \div \frac{1}{6}$ .

25. (a) State and solve a word problem whose solution is the quotient  $\frac{1}{2} \div \frac{5}{6}$ .
- (b) State and solve a word problem whose solution is the quotient  $\frac{5}{6} \div \frac{1}{2}$ .

- (c) The answer to the above division problem is  $1\frac{2}{3}$ . What does the  $\frac{2}{3}$  mean in the problem you stated above?
26. Show that the commutative property does not hold for division.
27. Show that the associative property does not hold for division.
28. Explain to Max why  $\frac{11}{4}$  is equal to  $11 \div 4$ .
29. (a) Explain to a student, in plain English and no mathematical manipulation, why  $5 \div \frac{2}{3}$  is bigger than 5.
- (b) We can make a general statement here: If  $\frac{a}{b}$  is a positive rational number less than 1, then  $5 \div \frac{a}{b} > 5$ . Prove this statement.
- (c) Can you use the same explanation from part (a) to explain why  $\frac{1}{5} \div \frac{2}{3}$  is bigger than  $\frac{1}{5}$ . Does this seem to be harder? Why or why not?
- (d) Prove that if  $\frac{a}{b}$  is a positive rational number less than 1, then  $\frac{1}{5} \div \frac{a}{b} > \frac{1}{5}$ .
30. Recall that the fundamental law of fractions says that  $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$  for any integer  $n$ . So, for example,  $\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$ . We want to explore whether  $n$  must be an integer.
- (a) Choose integers for  $a$  and  $b$ . Now choose a rational number between 0 and 1 for  $n$ . Is  $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ ?
- (b) Choose integers for  $a$  and  $b$ . Now choose a rational number (that is not an integer) greater than 1 for  $n$ . Is  $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ ?
- (c) Your answer to the previous two problems should be yes. Therefore it appears the Fundamental Law holds for rational multipliers not just integers. Show that  $\frac{a}{b} = \frac{a \cdot \frac{c}{d}}{b \cdot \frac{c}{d}}$ . Be sure to justify all equalities.

## 2 Decimals and Percents

### 2.1 Problems

- Use the definition of decimals to express the following numbers as a fraction in simplest form.
  - .36
  - .02918
- Use multiplication by powers of ten to express the following numbers as a fraction in simplest form.
  - .36
  - .02918
- Without dividing, express the following rational numbers as a decimal. Be sure to show all steps.
  - $\frac{3}{4}$
  - $\frac{9}{40}$
- Write the following decimals as fractions in simplest form.
  - $\overline{.4}$
  - $\overline{.12}$
  - $\overline{.319}$
- Consider the fraction  $\frac{2}{13}$ .
  - Without dividing, explain how you know this fraction can be expressed as a repeating decimal.
  - Before actually finding the decimal, determine what the longest the repetend could be? Explain carefully.
  - Express  $\frac{2}{13}$  as a decimal. Be sure to show all your steps.
- In a plethora of math books you will see the statement  $1 = \overline{.9}$ .
  - Using the algorithm we discussed in class, convert this repeating decimal to a fraction in simplest form. (Note you should get 1.)

- (b) What do we mean when we say 1 is equal to  $\overline{.9}$ ?
7. The definition of  $.123$  is one-tenth plus two hundredths plus three thousandths. However Max hears you call  $.123$  one hundred twenty three thousandths. Explain to Max why you call it that.
  8. Max says to you that he thinks  $.36 > .4$  because  $36 > 4$ . How would you respond to him?
  9. Max hears you say that  $.3$  is the same as  $.30$ . This confuses him however, because 3 is not the same as 30. He asks why putting zeros at the end of a decimal doesn't change the number, but putting zeros at the end of a whole number does? How would you respond?
  10. In class we made the following statement:

Let  $\frac{a}{b}$  be a rational number in simplest form. Then  $\frac{a}{b}$  can be written as a terminating decimal if and only if the prime factorization of  $b$  contains no primes other than 2 and 5.

If we remove the assumption that the fraction is in simplest form, then the statement is false. Give an example to show this.

11. (a) Max asks, "Why do we have to line up the decimal when we are adding but not when we are multiplying?" How do you respond to Max?
  - (b) He then asks, "Will I get the wrong answer if I line up the decimals when multiplying?" How do you respond to Max?
12. (a) Give your students a step by step guide on how to subtract two decimals.
  - (b) Do the problem  $25.6 - 4.712$ .
  - (c) Use the above example to show why the algorithm you gave in part 12a works.
13. (a) Give your students a step by step guide on how to multiply two decimals.
  - (b) Do the problem  $23.4 \times 2.568$ .
  - (c) Use the above example to show why the algorithm you gave in part 13a works.

14. (a) Give your students a step by step guide on how to divide a whole number into a decimal.  
(b) Do the problem  $54.18 \div 5$ .  
(c) Use the above example to show why the algorithm you gave in part 14a works.
15. (a) Give your students a step by step guide on how to divide a decimal into a decimal.  
(b) Do the problem  $54.18 \div 2.5$ .  
(c) Use the above example to show why the algorithm you gave in part 15a works.
16. A baseball is on sale for 20% off. The regular price of the ball is \$6.75. Let  $x$  = sale price of the ball.  
(a) Write a proportion relating 20, 6.75 and  $x$ . Solve for  $x$ .  
(b) Using the fact that  $20\% = .20$ , write an equation (that is not a proportion) relating .20, 6.75 and  $x$ . Solve for  $x$ .  
(c) What is the sale price of the ball?
17. A baseball is on sale for 20% off, which makes the sale price \$5.40. Let  $x$  = the regular price of the baseball.  
(a) Write a proportion relating 20, 5.40 and  $x$ . Solve for  $x$ .  
(b) Using the fact that  $20\% = .20$ , write an equation (that is not a proportion) relating .20, 5.40 and  $x$ . Solve for  $x$ .  
(c) What is the regular price of the ball?
18. The regular price of a baseball is \$6.75 and the sale price is \$5.40.  
(a) Let  $x$  = the percentage off the regular price. Write a proportion relating 6.75, 5.40 and  $x$ . Solve for  $x$ .  
(b) Let  $x$  = the decimal equivalent of the percentage off the regular price. Write an equation (that is not a proportion) relating 6.75, 5.40 and  $x$ . Solve for  $x$ .  
(c) What is the percentage off?
19. Sally's tuition went up 15% at the end of her first year. At the end of her second year, tuition went up 15% again. Sally says the total hike in tuition after two years is 30% of her original tuition. Is Sally correct? If so, prove it. If not, give a counterexample.

20. There are 40 students in Mr. Jones' class and 60 students in Ms. Frank's class. Max took a survey of the students to find out how popular peas are. He found that 20% of the students in Mr. Jones' class don't like peas and 25% of Ms. Frank's class don't like peas. Max told Mr. Jones' and Ms. Frank that 45% of the students don't like peas. Is Max correct? If yes, show it. If not, what percentage of the students do not like peas?
21. Max also took a survey on favorite colors. He told the students they could only pick one favorite color. Max found that 20% of the students in Mr. Jones' class have favorite color red, while 35% have favorite color blue. Max concludes that 55% of Mr. Jones class have favorite color red or blue. Is Max correct? If yes, show it. If not, what percentage of the students have favorite color red or blue?
22. Max surveyed residents in Snaketown. They found that 30% of the residents are employed in town, 50% are employed outside of town and 25% are unemployed. Max said the results cannot possibly be correct because they add up to more than 100%, but Molly said she doesn't think that's a problem. Why would Max think these results don't make sense? Why would Molly think the results are fine?
23. In Coldtown, there are several fines residents must be prepared to pay. They are all based on your monthly salary. For example, the fine for leaving finger prints on store windows is 20% of your monthly salary, the fine for being noisy after 10PM is 50% of your monthly salary and the fine for not recycling is 40% of your monthly salary. If you are caught doing all three of these things in one month, then you would owe 110% of your monthly salary. Max says that this is not possible because 110% doesn't make sense. Is Max correct or not? Explain.

### 3 Irrational Numbers

#### 3.1 Problems

1. Determine which of the following are rational numbers. Be sure to show all work.

(a)  $\sqrt{\frac{144}{169}}$

(b)  $\sqrt{175}$

(c)  $\sqrt{.2\bar{5}}$

(d)  $\sqrt{1.\bar{7}}$

2. You have just taught Max the definition of  $\sqrt{\quad}$ , but he has several questions. (Max is not allowed to use calculators, so your explanation cannot rely on use of a calculator.)

(a) Since  $(-3)^2 = 9$ , why doesn't  $\sqrt{9} = -3$ ?

(b) Does  $\sqrt{4} + \sqrt{5} = \sqrt{4+5}$  ?

(c) Does  $\sqrt{4}\sqrt{5} = \sqrt{4 \cdot 5}$  ?

(d) Why can't you take the square root of a negative number?

3. In light of what you explained to Max in problem 2d, Max now believes that you cannot solve the equation  $\sqrt{-x} = 9$ . Is Max correct? Explain.

4. Ms. Frank (one of your colleagues) uses her calculator to compute  $\sqrt{13}$ . The calculator spits out the answer 3.60555.

(a) Ms. Frank proceeds to tell her class that  $\sqrt{13} = 3.60555$ . Explain to Ms. Frank why  $\sqrt{13}$  cannot possibly be represented as a terminating decimal.

(b) After your explanation above, Ms. Frank then concludes that  $\sqrt{13} = 3.60\bar{5}$ . Explain to Ms. Frank why  $\sqrt{13}$  cannot possibly be represented as a repeating decimal.

5. Give an example for each of the following.

(a) A natural number.

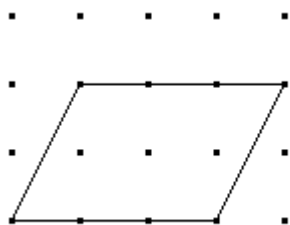
- (b) A whole number that is not a natural number.
  - (c) An integer that is not a whole number.
  - (d) A rational number that is not an integer.
  - (e) A real number that is not a rational number.
6. Is it possible to answer any of the questions from problem 5(b)-(e) in the reverse? For example, a rational number that is not a real number or an integer that is not a rational number, etc. If yes, give an example. If no, explain why not.
7. We proved in class that  $\sqrt{7}$  is an irrational number. Using this fact, prove the following statements.
- (a)  $2 - \sqrt{7}$  is irrational.
  - (b)  $6\sqrt{7}$  is irrational.
8. Consider the numbers  $.5\overline{3}$  and  $.5\overline{4}$ .
- (a) Find a rational number between these two numbers.
  - (b) Find an irrational number between these two numbers.

## 4 Geometry

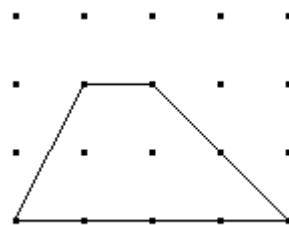
### 4.1 Problems

- Using the fact that the vertical and horizontal distance between consecutive dots is 1 unit, find the area of the shapes drawn on dot paper below. Be sure to show all work clearly (colored pencils may be useful). You may not use any area formulas.

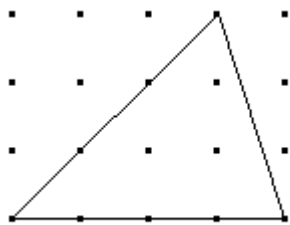
(a)



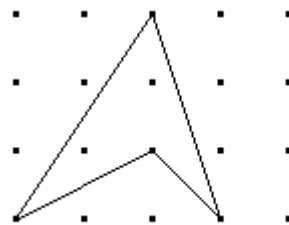
(b)



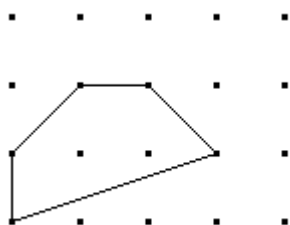
(c)



(d)

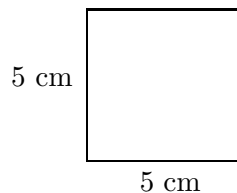


(e)

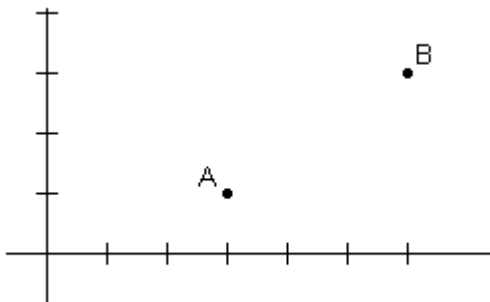


2. In 1899, G. Pick discovered a very surprising theorem. We'll use the shapes above to explore his theorem.
- Let  $I$  equal the number of dots inside the shape. Find  $I$  for each of the shapes above.
  - Let  $B$  equal the number of dots that lie on the shape. Find  $B$  for each of the shapes above.
  - Compute  $I + \frac{1}{2}B - 1$  for each of the shapes above.
  - Your answer from part (c) should match up with the areas you computed in problem 1. Do they? That is precisely what Pick's theorem says: The area of a shape constructed on dot paper is equal to  $I + \frac{1}{2}B - 1$ . Draw a shape of your choosing on dot paper. Compute the area manually as you did in problem 1, then check that you get the same result using Pick's Theorem.
3. (a) Find the dimensions of a square that has area  $6.25 \text{ in}^2$ .  
 (b) Explain why it is impossible to construct a square on dot paper with area  $6.25$  square units.
4. Give the dimensions of a rectangle that satisfy the following.  $A$  represents area and  $P$  represents perimeter. Round your answers to the nearest hundredth.
- $A = 40 \text{ in}^2$     $P = 30 \text{ in}$
  - $A = 25 \text{ in}^2$     $P = 30 \text{ in}$
5. Max is comparing two rectangles and says that the rectangle with the larger perimeter must have larger area. Will Max always be correct? If so, explain why. If not, give Max an example to see that his conclusion may be false.
6. For each of the questions below an example is not enough, you must show that your answer holds for *any* rectangle.
- If you triple the lengths of all the sides in a rectangle, what is the change in the perimeter?
  - If you triple the lengths of all the sides in a rectangle, what is the change in the area?

7. Max interpreted  $5 \text{ cm}^2$  as shown below. Explain to Max why he is incorrect, and show him how to correctly interpret  $5 \text{ cm}^2$ .



8. Starting from point A, a boat sails due south for 6 miles, then due east for 5 miles and then due south for 4 miles.
- (a) Draw a picture to represent this situation.
  - (b) How far is the boat from point A at the end of this journey?
9. (a) Max has three sticks. One of length 2 inches, one of length 3 inches and one of length 4 inches. He wants to make a right triangle. Explain to Max why this is impossible.
- (b) Molly has the same three sticks. After hearing your explanation from above, Max tells Molly that she can't make a triangle with the sticks. Is he correct?
  - (c) Mason has three sticks. One of length 2 inches, one of length 3 inches and one of length 5 inches. After hearing your explanation from above, Max tells Mason that she could make a triangle it just wouldn't be a right triangle. Is Max correct?
10. Use the Pythagorean theorem to find the distance between points *A* and *B*. **DO NOT** simply apply the distance formula. Be sure to give a careful explanation.



11. Max is painting 1 foot by 1 foot tiles for his grandma. One ounce of paint will cover 100 square inches. He wants to paint the shaded parts blue and the rest will be painted red. How much paint of each color does Max need? (All curves are circles or semicircles.)

