HW #1 – ch. 1 problems 23-27, 29 – SOLUTIONS

23. Recall that we defined negative exponents in class. We said that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

    (a) Using this definition, explain to Max how to compute 2^{-3}.

    $$2^{-3} = \left(\frac{2}{1}\right)^{-3} = \left(\frac{1}{2}\right)^{3} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

    (b) If a is an integer, prove that $a^{-n} = \frac{1}{a^n}$.

    $$a^{-n} \overset{(1)}{=} \left(\frac{a}{1}\right)^{-n} \overset{(2)}{=} \left(\frac{1}{a}\right)^{n} \overset{(3)}{=} \underbrace{\frac{1}{a}\cdot\frac{1}{a}\cdot\dots\cdot\frac{1}{a}}_{n \text{ times}} \overset{(4)}{=} \frac{1}{a^n}$$

    Justifications:

    (1) Integer in rational form
    (2) Definition of negative exponents
    (3) Definition of natural number exponents
    (4) Multiplication in $\mathbb{Q}$

    (c) Max says that since $a^{-n} = \frac{1}{a^n}$, then $(a+b)^{-n} = \frac{1}{a^n} + \frac{1}{b^n}$. Is Max correct? If yes, prove it. If not, give Max a counterexample and show him what $(a+b)^{-n}$ does equal.
    No, Max is not correct. See the example below.

    $$(2+3)^{-1} = 5^{-1} = \frac{1}{5}$$

    $$\frac{1}{2^1} + \frac{1}{3^1} = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

    Since $a+b$ is in the paranthesis, $(a+b)^{-n} = \frac{1}{(a+b)^n}$, which is not equal to $\frac{1}{a^n} + \frac{1}{b^n}$.

26. Show that the commutative property does not hold for division.

    $$1 \div 2 = \frac{1}{2} \neq 2 = 2 \div 1$$

27. Show that the associative property does not hold for division.

    $$8 \div (4 \div 2) = 8 \div 2 = 4$$

    $$(8 \div 4) \div 2 = 2 \div 2 = 1$$

    Thus $8 \div (4 \div 2) \neq (8 \div 4) \div 2$.

29. (a) Explain to a student, in plain English and no mathematical manipulation, why $5 \div \frac{2}{3}$ is bigger than 5.

    $5 \div \frac{2}{3}$ is asking how many times $\frac{2}{3}$ goes into 5. We know 1 goes into 5 five times. Since two thirds is less than 1 it will go into 5 more than five times. Therefore $5 \div \frac{2}{3}$ is larger than 5.

(b) We can make a general statement here: If $\frac{a}{b}$ is a positive rational number less than 1, then $5 \div \frac{a}{b} > 5$. Prove this statement.

We need to compare $5 \div \frac{a}{b}$ and 5, so I will compute $5 \div \frac{a}{b}$ and compare it to 5 by getting a common denominator.

$$5 \div \frac{a}{b} = 5 \cdot \frac{b}{a} = \frac{5}{1} \cdot \frac{b}{a} = \frac{5b}{a}$$

$$5 = \frac{5}{1} = \frac{5a}{a}$$

Since we know that $\frac{a}{b}$ is between 0 and 1, then $b > a$. So $5b > 5a$ and thus $\frac{5b}{a} > \frac{5a}{a}$. In other words, $5 \div \frac{a}{b} > 5$.

(c) Can you use the same explanation from part (a) to explain why $\frac{1}{5} \div \frac{2}{3}$ is bigger than $\frac{1}{5}$. Does this seem to be harder? Why or why not?

Yes it is harder because we are no longer working with a whole number. Moreover two thirds is larger than one fifth so thinking how many two thirds go into one fifth is not very intuitive.

(d) Prove that if $\frac{a}{b}$ is a positive rational number less than 1, then $\frac{1}{5} \div \frac{a}{b} > \frac{1}{5}$.

We need to compare $\frac{1}{5} \div \frac{a}{b}$ and $\frac{1}{5}$, so I will compute $\frac{1}{5} \div \frac{a}{b}$ and compare it to $\frac{1}{5}$ by getting a common denominator.

$$\frac{1}{5} \div \frac{a}{b} = \frac{1}{5} \cdot \frac{b}{a} = \frac{b}{5a}$$

$$\frac{1}{5} = \frac{a}{5a}$$

Since we know that $\frac{a}{b}$ is between 0 and 1, then $b > a$. Thus $\frac{b}{5a} > \frac{a}{5a}$. In other words, $\frac{1}{5} \div \frac{a}{b} > \frac{1}{5}$.