Author:______________________________ **Understanding Multiplication**

Group Members:

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1. Each member in your group choose a rational number between 0 and 1 (not including 0 and 1).

   (a) Compute the product of your number with $\frac{2}{5}$.

   (b) Is the product greater than or less than $\frac{2}{5}$.

   (c) After comparing results with other members in your group, state a conjecture that tells which is bigger, $\dfrac{2}{5}$ or $\dfrac{2}{5} \cdot \dfrac{a}{b}$,

   Let $\dfrac{a}{b}$ be a rational number between 0 and 1, then $\dfrac{2}{5} \cdot \dfrac{a}{b}$ is less than $\dfrac{2}{5}$.

   (d) Prove your conjecture.

   $\dfrac{2}{5} \cdot \dfrac{a}{b} = \dfrac{2a}{5b}$ (Multiplication in $\mathbb{Q}$)

   $\dfrac{2}{5} = \dfrac{2b}{5b}$ (Fundamental Law)

   Since $\dfrac{a}{b}$ is between 0 and 1, we know that a is less than b. Therefore $2a$ is less than $2b$ and hence $\dfrac{2a}{5b}$ is less than $\dfrac{2b}{5b}$. Therefore $\dfrac{2}{5} \cdot \dfrac{a}{b}$ is less than $\dfrac{2}{5}$.

2. Two students were given the following problem. Max and Mary each have 12 strips of ribbon. Max's strips are each $2\dfrac{1}{4}$ feet long, while Mary's are each $1\dfrac{5}{8}$ feet long. How much more ribbon does Max have than Mary?

   - Student 1 solved it in the following way.
     First find the difference in length between the strips:
     $2\dfrac{1}{4} - 1\dfrac{5}{8} = \dfrac{9}{4} - \dfrac{13}{8} = \dfrac{18}{8} - \dfrac{13}{8} = \dfrac{5}{8}$
     Now multiply that by 12:
     $12 \cdot \dfrac{5}{8} = \dfrac{60}{8} = \dfrac{15}{2} = 7\dfrac{1}{2}$
     Therefore Max has seven and a half feet more ribbon than Mary.

   - Student 2 solved it in the following way.
     Find how much ribbon Max has:
     $12 \cdot 2\dfrac{1}{4} = 12 \cdot \dfrac{9}{4} = \dfrac{108}{4} = 27$
     Find how much ribbon Mary has:
     $12 \cdot 1\dfrac{5}{8} = 12 \cdot \dfrac{13}{8} = \dfrac{156}{8} = \dfrac{39}{2} = 19\dfrac{1}{2}$
     Find how much more Max has than Mary:
     $27 - 19\dfrac{1}{2} = \dfrac{54}{2} - \dfrac{39}{2} = \dfrac{15}{2} = 7\dfrac{1}{2}$
     Therefore Max has seven and a half feet more ribbon than Mary.

(a) Which way do you think is better?

(b) What algebraic property insures both methods are correct? Explain.

If you combine the steps into one, Student 1 did $12\left(2\frac{1}{4} - 1\frac{5}{8}\right)$ where as Student 2 did $12 \cdot 2\frac{1}{4} - 12 \cdot 1\frac{5}{8}$. Thus the distributive property of multiplication over addition insures both methods will always give the same answer.

3. Recall that the definition of (natural number) exponents says that a^n is equal to a multiplied by itself n times. Simplify the following expressions. You may not use any exponent rules (other than the definition of exponents).

(a) $a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5$

(b) $\dfrac{a^5}{a^2} = \dfrac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = \dfrac{a \cdot a \cdot a}{1} = a \cdot a \cdot a = a^3$

(c) $(a^2)^3 = (a \cdot a)^3 = (a \cdot a)(a \cdot a)(a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$

4. Consider the examples above and explain why the following exponent rules hold.

(a) $a^n a^m = a^{n+m}$

According to the definition of exponents a^n means we have a total of n a's multiplied. Similarly we also have m a's multiplied. This gives us a total of $m + n$ a's multiplied. See below.

$$a^n a^m = (\underbrace{a \cdots a}_{n \text{ times}})(\underbrace{a \cdots a}_{m \text{ times}}) = \underbrace{a \cdots\cdots a}_{n+m \text{ times}} = a^{n+m}$$

(b) $\dfrac{a^n}{a^m} = a^{n-m}$

There are n a's multiplied on top and m a's multiplied on bottom. Using fundament law we can cancel the m a's on the bottom with m a's on top. This leaves $n - m$ a's on top. Note: Right now we've explained this rule only if n is bigger than m. When we get to negative exponents we will discuss this again.

(c) $(a^n)^m = a^{nm}$

In this one we have n a's multiplied by itself m times. Which gives us a total of nm a's. See below.

$$(a^n)^m = (\underbrace{a \cdots a}_{n \text{ times}})^m = \underbrace{(\underbrace{a \cdots a}_{n \text{ times}}) \cdots (\underbrace{a \cdots a}_{n \text{ times}})}_{m \text{ times}} = \underbrace{a \cdots\cdots\cdots a}_{nm \text{ times}} = a^{nm}$$

5. Prove the following statements. Be sure to justify each equality.

(a) For all natural numbers a and b, $\left(\dfrac{a}{b}\right)^4 = \dfrac{a^4}{b^4}$.

$$\left(\frac{a}{b}\right)^4 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a \cdot a \cdot a \cdot a}{b \cdot b \cdot b \cdot b} = \frac{a^4}{b^4}$$

The justification for equalities:

i. Definition of natural number exponents
ii. Multiplication in $\mathbb{Q}$
iii. Definition of natural number exponents

(b) For all rational numbers a and b, $(ab)^4 = a^4 b^4$.

$(ab)^4 = (ab)(ab)(ab)(ab) = (aaaa)(bbbb) = a^4 b^4$

The justification for equalities:

i. Definition of natural number exponents

ii. Commutative and Associative properties of multiplication

iii. Definition of natural number exponents

6. Give a counterexample to show that the following statements are false.

(a) For all rational numbers a and b and natural number n, $a^m \cdot a^n = a^{mn}$.

Let $a = 2, m = 1, n = 3$.

$a^m \cdot a^n = 2^1 \cdot 2^3 = 2 \cdot 8 = 16$

$a^{nm} = 2^{1 \cdot 3} = 2^3 = 8 \neq 16$

Thus $a^m \cdot a^n = a^{nm}$ is not true for all a, m and n.

(b) For all rational numbers a and b and natural number n, $(a + b)^n = a^n + b^n$.

Let $a = 2, b = 1, n = 3$.

$(a + b)^n = (2 + 1)^3 = 3^3 = 27$

$a^n + b^n = 2^3 + 1^3 = 8 + 1 = 9 \neq 27$

Thus $(a + b)^n = a^n + b^n$ is not true for all a, b and n.