Ring Theory Definitions

1 Definitions

Definition 1. *A* **ring***, R, is a set with two binary operations, addition and multiplication, such that*

1. $(R, +)$ *is an abelian group (additive identity is usually denoted* 0 *or* 0_R*)*
2. $a(bc) = (ab)c$ *for all* $a, b, c \in R$*, and*
3. $a(b + c) = ab + ac$ *and* $(b + c)a = ba + ca$ *for all* $a, b, c \in R$*.*

Definition 2. *If a ring, R, contains an element, 1, such that $a1 = a = 1a$ for all $a \in R$, we say that R is a* **ring with unity***.*

Definition 3. *If $ab = ba$ for all elements a and b in a ring, we say that the ring is* **commutative***.*

Definition 4. *In a ring with unity, an element with a multiplicative inverse is called a* **unit***. That is, $a \in R$ is a unit if there exists $b \in R$ such that $ab = 1 = ba$.*

Definition 5. *In a ring, R, if $ab = 0$ but $a \neq 0$ and $b \neq 0$, then a and b are said to be* **zero divisors***. The set of all zero-divisors of R is denoted $ZD(R)$.*

Definition 6. *An* **integral domain** *is a commutative ring with unity $1 \neq 0$ with no zero-divisors.*

Definition 7. *A commutative ring with unity $1 \neq 0$ is a* **field** *if every nonzero element is a unit.*

Definition 8. *A ring with unity $1 \neq 0$ in which every nonzero element is a unit is called a* **division ring***.*

Definition 9. *If R is a ring and $S \subseteq R$ and S is a ring under the operations of R restricted to S, then S is said to be a* **subring** *of R.*

Theorem 10. *If R is a ring and $S \subseteq R$, then S is a subring of R if and only if the following hold.*

1. $S \neq \emptyset$
2. $a - b \in S$ *for all* $a, b \in S$
3. $ab \in S$ *for all* $a, b \in S$

2 Exercises

1. Read and study section 3.2 on your own.

2. Problems #3,5,6,7,8,10,12,13 in book. (I will collect problem #12 on Monday, November 16.)

3. Prove that in a ring with unity, $U(R) \cap ZD(R) = \emptyset$.

4. Let R be a ring and let $a \in R$. Prove that if $a \neq 0$ and $a \notin ZD(R)$, then $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$ for all $b, c \in R$.

5. Prove that every finite integral domain is a field.

6. Let R be a ring in which $x^2 = x$ for all $x \in R$. (Such a ring is called a Boolean ring.) Prove that R is a commutative ring.

7. Find all subrings of $\mathbb{Z}$. (Compare to problem #9 in book.)

8. Find an example of a ring with elements a and b such that a and b are zero divisors, but $a + b \neq 0$ and $a + b$ is not a zero divisor.