

The Sylow Theorems

1 Exercises

1. From the text (section 5.2): # 23, 27, 28, 29, 30, 31
2. Prove the following
 - (a) Every group of order 15 is Abelian.
 - (b) There are no more than 4 non-isomorphic groups of order 30.
 - (c) There are at least 4 non-isomorphic groups of order 30. (Describe them in terms of groups that we know and explain how you know that the four you've described are non-isomorphic.)
3. Let G be a group of order 48. Show that the intersection of any two distinct Sylow 2-subgroups of G has order 8.
4. Let G be a group with $|G| = 56$. Prove that G is not simple.
5. What is the smallest composite integer n such that there is a unique group of order n ?
6. Let G be a noncyclic group of order 21.
 - (a) How many 3-Sylow subgroups does G .
 - (b) Prove that G has 14 elements of order 3.
7. Let G be a group of order 60. Show that G has exactly four elements of order 5 or exactly 24 elements of order 5. Which of these cases holds for A_5 ?
8. Let G be a group of order 60 and let $H \triangleleft G$ with $|H| = 2$. Show
 - (a) G has normal subgroups of order 6, 10 and 30,
 - (b) G has subgroups of order 12 and 20, and
 - (c) G has a cyclic subgroup of order 30.
9. Let G be a group of order 60. If the Sylow 3-subgroup is normal, show that the Sylow 5-subgroup is also normal.
10. Let $|G| = 7^2 \cdot 13$. Prove G is Abelian.

A group G is said to be **solvable** if there exist subgroups $G_0, G_1, G_2, \dots, G_k$ such that

$$\{e\} = G_k \triangleleft G_{k-1} \triangleleft \cdots \triangleleft G_2 \triangleleft G_1 \triangleleft G_0 = G \quad (\star)$$

and such that G_i/G_{i+1} is Abelian for all i . This sequence of subgroups, \star , is called a **solvable series** for G .

11. Prove that S_4 is solvable.
12. Prove that if G is solvable and $H \leq G$, then H is solvable.
13. Suppose G is solvable and $\phi : G \rightarrow \overline{G}$ is a homomorphism from G to \overline{G} . Prove $\phi(G)$ is solvable.
14. Let G be a group and $H \triangleleft G$. Suppose H and G/H are both solvable. Prove G is solvable.
15. Let G be a group with $H \leq G$ and $K \triangleleft G$. Prove that if H and K are both solvable, then HK is solvable.
16. Let G be a group of order $495 = 3^2 \cdot 5 \cdot 11$.
 - a. What are the possible numbers of Sylow subgroups? (i.e. what are the possibilities for n_3 , n_5 and n_{11} ?)
 - b. Prove that a 5-Sylow subgroup or a 11-Sylow subgroup is normal.
 - c. Let K be the normal subgroup from part (b). Prove G/K is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_m \times \mathbb{Z}_9$ where $m \in \{5, 11\}$.
 - d. Let H_5 be a 5-Sylow subgroup of G and let H_{11} be an 11-Sylow subgroup of G . Prove $H_5H_{11} \triangleleft G$. (part c might be helpful – one of these two subgroups is the K in part c).
 - e. Find a solvable series for G .