

Direct Products

Let G be a group with $H \leq G$ and $K \leq G$. Define $HK = \{hk | h \in H, k \in K\}$.

1. Prove that the following are equivalent.
 - (a) $HK \leq G$
 - (b) HK is closed under the binary operation of G restricted to HK
 - (c) $HK = KH$
2. Prove that if $H \triangleleft G$ or $K \triangleleft G$, then $HK \leq G$.
3. Prove that if $H \triangleleft G$ and $K \triangleleft G$, then $HK \triangleleft G$.
4. Suppose H and K are finite subgroups of G .
 - (a) For each $x = hk \in HK$, define $\phi : H \cap K \rightarrow \{(h', k') \in H \times K | h'k' = x\}$ by $\phi(d) = (hd, d^{-1}k)$. Prove that ϕ is a bijection.
 - (b) Prove that if $H \cap K = \{e\}$, then each element of HK can be expressed uniquely as a product hk where $h \in H$ and $k \in K$.
 - (c) Prove $|HK| = \frac{|H||K|}{|H \cap K|}$.

Definition 1. Let G_1 and G_2 be groups. The **(external) direct product** of G_1 and G_2 is $G_1 \times G_2 = \{(g_1, g_2) | g_1 \in G_1, g_2 \in G_2\}$ and is a group under the binary operation $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$ where the operation in the first component is the operation of G_1 and the operation in the second component is the operation of G_2 .

Definition 2. Let G be a group with $H \triangleleft G$ and $K \triangleleft G$ such that $G = HK$ and $H \cap K = \{e\}$, then G is said to be the **internal direct product** of H and K .

Lemma 3. If G is a group with $H \triangleleft G$ and $K \triangleleft G$ such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$.

Proof. Assume $H \triangleleft G$, $K \triangleleft G$ and $H \cap K = \{e\}$. Let $h \in H$ and $k \in K$. So $h^{-1} \in H$. Now, since both subgroups are normal we have that $kh^{-1}k^{-1} \in H$ and $hkh^{-1} \in K$. Since $kh^{-1}k^{-1} \in H$ and $h \in H$, then $hkh^{-1}k^{-1} \in H$. Similarly since $hkh^{-1} \in K$ and $k^{-1} \in K$, then $hkh^{-1}k^{-1} \in K$. Therefore $hkh^{-1}k^{-1} \in H \cap K$. So $hkh^{-1}k^{-1} = e$ and hence $hk = kh$. \square

Theorem 4. If G is the internal direct product of H and K , then $HK \cong H \times K$.

Proof. Assume G is the internal direct product of H and K . So $H \triangleleft G$, $K \triangleleft G$ and $H \cap K = \{e\}$. Define $\phi : H \times K \rightarrow HK$ by $\phi(h, k) = hk$. We will first show ϕ is a homomorphism. Let

$(h_1, k_1), (h_2, k_2) \in H \times K$. Then we have the following.

$$\begin{aligned}\phi((h_1, k_1)(h_2, k_2)) &= \phi(h_1h_2, k_1k_2) \\ &= h_1h_2k_1k_2 \\ &= h_1k_1h_2k_2 \text{ (by lemma above)} \\ &= \phi(h_1, k_1)\phi(h_2, k_2)\end{aligned}$$

Therefore ϕ is a homomorphism. Clearly ϕ is onto. Now assume $\phi(h_1, k_1) = \phi(h_2, k_2)$, so $h_1k_1 = h_2k_2$. Since $H \cap K = \{e\}$, then problem 4b above implies each element can be expressed uniquely in HK . Thus $h_1k_1 = h_2k_2$ implies $h_1 = h_2$ and $k_1 = k_2$. Hence ϕ is one-to-one. Therefore ϕ is an isomorphism and $H \times K \cong HK$. \square

Theorem 5. *If $G \cong G_1 \times G_2$, then there exist normal subgroups H_1 and H_2 of G such that G is the internal direct product of H_1 and H_2 .*

Proof. Assume $G \cong G_1 \times G_2$. So there exists an isomorphism $\phi : G_1 \times G_2 \rightarrow G$. Define the following.

$$\begin{aligned}H_1 &= \phi(G_1 \times \{e\}) \\ H_2 &= \phi(\{e\} \times G_2)\end{aligned}$$

By previous homeworks we know that $H_1 \leq G$ and $H_2 \leq G$. Moreover, since $G_1 \times \{e\} \triangleleft G_1 \times G_2$ and $\{e\} \times G_2 \triangleleft G_1 \times G_2$, one can easily show that $H_1 \triangleleft G$ and $H_2 \triangleleft G$.

Now we will show that $H_1 \cap H_2 = \{e\}$. Let $x \in H_1 \cap H_2$. Since $x \in H_1$ there exists $(a, e) \in G_1 \times \{e\}$ such that $\phi(a, e) = x$. Similarly, since $x \in H_2$ there exists $(e, b) \in \{e\} \times G_2$ such that $\phi(e, b) = x$. Therefore $\phi(a, e) = \phi(e, b)$. However, ϕ is one-to-one, hence $(a, e) = (e, b)$ and thus $a = e$ and $b = e$. Therefore $x = \phi(e, e) = e$. So $H_1 \cap H_2 = \{e\}$.

Finally we will show that $G = H_1H_2$. Let $g \in G$. Since ϕ is onto there exists $(a, b) \in G_1 \times G_2$ such that $\phi(a, b) = g$. Therefore we have that $g = \phi(a, b) = \phi(a, e)\phi(e, b) \in H_1H_2$. Thus $G = H_1H_2$ and hence G is the internal direct product of H_1 and H_2 . \square