

28. (a) Let A be an $m \times n$ matrix with a row consisting entirely of zeros. Show that if B is an $n \times p$ matrix, then AB has a row of zeros.

Proof. Let $A = [a_{ij}]$ and assume that the l th row of A is all zeros. So, $a_{lj} = 0$ for all j between 1 and n . Let $B = [b_{ij}]$. Then $AB = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$. We claim that the l th row of AB is zero, so we need to show that $c_{lj} = 0$ for all j between 1 and n .

$$c_{lj} = \sum_{k=1}^n a_{lk}b_{kj} = \sum_{k=1}^n (0)b_{kj} = 0$$

Thus, the l th row of AB is zeros. □

- (b) Let A be an $m \times n$ matrix with a column consisting entirely of zeros and let B be $p \times m$. Show that BA has a column of zeros.

Proof. Let $A = [a_{ij}]$ and assume that the l th column of A is all zeros. So, $a_{il} = 0$ for all i between 1 and n . Let $B = [b_{ij}]$. Then $BA = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^n b_{ik}a_{kj}$. We claim that the l th column of BA is zero, so we need to show that $c_{il} = 0$ for all i between 1 and n .

$$c_{il} = \sum_{k=1}^n b_{ik}a_{kl} = \sum_{k=1}^n b_{ik}(0) = 0$$

Thus, the l th column of BA is zeros. □

51. Let \mathbf{x} be an n -vector.

- (a) Is it possible for $\mathbf{x} \cdot \mathbf{x}$ to be negative? Explain.

Let \mathbf{x} be an n -vector, so $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$. And since x_1, x_2, \dots, x_n are all

real numbers, then the squares of them are positive. So we are adding a bunch of non-negative numbers together and hence we must get a non-negative number.

- (b) If $\mathbf{x} \cdot \mathbf{x} = 0$, what is \mathbf{x} ?

Since the dot product shown above is $x_1^2 + x_2^2 + \cdots + x_n^2$, where each term is non-negative, there will never be any cancellation. Thus the only way for this sum to be zero is if each term is zero. Hence \mathbf{x} must equal the 0 vector.