16. Find a 2×2 matrix $B \neq 0$ and $B \neq I_2$ such that $AB = BA$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. How many such matrices B are there?

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$$

$a + 2c = a$ implies $c = 0$

$b + 2d = 2a + b$ implies $2d = 2a$ and hence $a = d$.

Thus for any matrix $B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, then $AB = BA$.

In particular, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. □

36. Suppose that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Solve the linear system $A\mathbf{x} = \mathbf{b}$ for each of the following matrices $\mathbf{b}$:

(a) $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

(b) $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 38 \\ 53 \end{bmatrix}$$

40. The linear system $C^T A\mathbf{x} = \mathbf{b}$ is such that A and C are nonsingular, with $A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

$$\mathbf{x} = A^{-1}(C^T)^{-1}\mathbf{b} = A^{-1}(C^{-1})^T\mathbf{b} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$