21. Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if $ad - bc \neq 0$. If this condition holds, show that

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Proof. We need to prove that if A is nonsingular, then $ad - bc \neq 0$. We also need to prove that if $ad - bc \neq 0$, then A is nonsingular.

Part 1 – Assume A is nonsingular, then A must be row equivalent to I_2. Let's perform elementary row operations on A.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$aR_2 - cR_1 \rightarrow R_2 \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Since the resulting matrix must be row equivalent to the identity, the bottom row cannot be all zeros. Therefore $ad - bc \neq 0$.

Part 2 – Assume $ad - bc \neq 0$. Since this quantity is non-zero, the matrix $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ is defined. By multiplying A with this matrix we can see that we get the identity.

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore the above matrix is A^{-1}, so A is nonsingular. □