10. Does the set $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ span M_{22}.

Let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix. We want to show there exist x, y, z, w such that the following equation holds.

$$x \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + w \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Therefore we have the following system:

$$\begin{aligned} x + z &= a_{11} \\ x + w &= a_{12} \\ y + w &= a_{21} \\ y + z + w &= a_{22} \end{aligned}$$

Therefore we have the following augmented matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & a_{11} \\ 1 & 0 & 0 & 1 & a_{12} \\ 0 & 1 & 0 & 1 & a_{21} \\ 0 & 1 & 1 & 1 & a_{22} \end{bmatrix}$$

$$\begin{matrix} \\ R_1 - R_2 \to R_2 \\ \\ R_4 - R_3 \to R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 & a_{11} \\ 0 & 0 & 1 & -1 & a_{11} - a_{12} \\ 0 & 1 & 0 & 1 & a_{21} \\ 0 & 0 & 1 & 0 & a_{22} - a_{21} \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 1 & 0 & a_{11} \\ 0 & 1 & 0 & 1 & a_{21} \\ 0 & 0 & 1 & 0 & a_{22} - a_{21} \\ 0 & 0 & 1 & -1 & a_{11} - a_{12} \end{bmatrix}$$

$$R_3 - R_4 \to R_4 \begin{bmatrix} 1 & 0 & 1 & 0 & a_{11} \\ 0 & 1 & 0 & 1 & a_{21} \\ 0 & 0 & 1 & 0 & a_{22} - a_{21} \\ 0 & 0 & 0 & 1 & a_{22} - a_{21} - a_{11} + a_{12} \end{bmatrix}$$

Therefore we can see that the above system of equations does have a solution. Therefore *span* $S = M_{22}$.

12. Find a set of vectors spanning the null space of $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$.

Let $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be a solution to the equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{r} \\ 2R_1 - R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 3R_2 + R_3 \to R_3 \\ 2R_2 - R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $-b - 2c = 0$ and $a + b + 2c - d = 0$, which gives $b = -2c$ and $a = -b - 2c + d = 2c - 2c + d = d$. Therefore we have the following:

$$\mathbf{x} = \begin{bmatrix} d \\ -2c \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -2c \\ c \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ 0 \\ 0 \\ d \end{bmatrix} = c \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus $S = \left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for the null space of A.