6. Show that the set of matrices $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ forms a basis for the vector space M_{22}.

$$a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a + c &= 0 \\ a + d &= 0 \\ b + d &= 0 \\ b + c + d &= 0 \end{aligned}$$

Subtracting the third and fourth equation gives $c = 0$. If $c = 0$, then from the first equation $a = 0$. Then by the second equation $d = 0$, and finally by the third equation $b = 0$. Therefore the set is linear independent.

Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be in M_{22}. We need to find a, b, c and d to solve the following equation.

$$a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$\begin{aligned} a + c &= x \\ a + d &= y \\ b + c &= z \\ b + c + d &= w \end{aligned}$$

Therefore we get the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 1(1(1)) + 1(1(1)) = 2$$

Since the determinant is non-zero, the above matrix equation has a solution. Therefore the span is M_{22}.

Therefore this set does form a basis of M_{22}.

14. Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$. Find a basis for the subspace $W = \text{span } S$.

Note that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Therefore we can throw those two vectors out. In other words, if $T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, then span T = span S. Moreover, T is linearly independent since they are not multiples of each other. Thus T is a basis for the span of S.

32. Find the dimension of the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $b = 3a - 5d$ and $c = d + 4a$.

$$\begin{aligned} at^3 + bt^2 + ct + d &= at^3 + (3a - 5d)t^2 + (d + 4a)t + d \\ &= a(t^3 + 3t^2 + 4t) + d(-5t^2 + t + 1) \end{aligned}$$

Therefore $S = \{t^3 + 3t^2 + 4t, -5t^2 + t + 1\}$ spans the vector space. We ned only show S is linearly independent.

$$\begin{aligned} r(t^3 + 3t^2 + 4t) + s(-5t^2 + t + 1) &= 0 \\ rt^3 + (3r - 5s)t^2 + (4r + s)t + s &= 0 \end{aligned}$$

By the t^3 coefficient, we can conclude $r = 0$ and by the constant coefficient we can conclude $s = 0$. Thus S is linearly independent. Hence S is a basis for the vector space.