16. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ be ordered bases for $\mathbb{R}^3$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 8 \\ 2 \end{bmatrix}$.

(a) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to the basis T.

We need to solve two linear systems here, so we will do it all at once by augmenting the matrix with both $\mathbf{v}$ and $\mathbf{w}$.

$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 3 & 8 \\ 0 & -1 & 0 & 8 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 & 7 \\ 0 & -1 & 0 & 8 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 & 7 \\ 0 & 0 & 1 & 28 & 13 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ 0 & 3 & 0 & -24 & -6 \\ 0 & 0 & 1 & 28 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & -27 & -3 \\ 0 & 3 & 0 & -24 & -6 \\ 0 & 0 & 1 & 28 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -9 & -1 \\ 0 & 1 & 0 & -8 & -2 \\ 0 & 0 & 1 & 28 & 13 \end{bmatrix}$$

$$[\mathbf{v}]_T = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}$$

$$[\mathbf{w}]_T = \begin{bmatrix} -1 \\ -2 \\ 13 \end{bmatrix}$$

(b) What is the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis?

We now have three linear systems to solve, namely find each of the basis vectors in T in terms of S. We will again augment the matrix with these three vectors.

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -6 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -2 & -5 & -2 \\ 0 & 1 & 0 & -1 & -6 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$$P_{S \leftarrow T} = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

(c) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to S, using $P_{S \leftarrow T}$.

$$\begin{aligned} [\mathbf{v}]_S &= \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} [\mathbf{v}]_T \\ &= \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [\mathbf{w}]_S &= \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} [\mathbf{w}]_T \\ &= \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 13 \end{bmatrix} \\ &= \begin{bmatrix} -14 \\ -13 \\ 8 \end{bmatrix} \end{aligned}$$

(d) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to S directly.

$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 8 \\ 1 & 0 & 2 & 8 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 1 & 2 & 7 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 1 & 3 & 8 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -13 \\ 0 & 0 & 1 & 3 & 8 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 2 & -14 \\ 0 & 1 & 0 & 1 & -13 \\ 0 & 0 & 1 & 3 & 8 \end{bmatrix}$$

$$[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$[\mathbf{w}]_S = \begin{bmatrix} -14 \\ -13 \\ 8 \end{bmatrix}$$

(e) Find the transition matrix $Q_{T \leftarrow S}$ from the S-basis to the T-basis.

$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 4 & -1 & 7 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 4 & -1 & 7 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 & 0 & -6 & 3 & -6 \\ 0 & 3 & 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 4 & -1 & 7 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 4 & -1 & 7 \end{bmatrix}$$

$$Q_{T \leftarrow S} = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}$$

(f) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to T, using $Q_{T \leftarrow S}$. Compare the answers with those of (a).

$$\begin{aligned} [\mathbf{v}]_T &= \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix} [\mathbf{v}]_S \\ &= \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [\mathbf{w}]_T &= \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix} [\mathbf{w}]_S \\ &= \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix} \begin{bmatrix} -14 \\ -13 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -2 \\ 13 \end{bmatrix} \end{aligned}$$

22. Let $S = \{[-1 \;\; 2 \;\; 1], [0 \;\; 1 \;\; 1], [-2 \;\; 2 \;\; 1]\}$ and $T = \{[-1 \;\; 1 \;\; 0], [0 \;\; 1 \;\; 0], [0 \;\; 1 \;\; 1]\}$ be ordered bases for $\mathbb{R}_3$, and $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Determine $[\mathbf{v}]_T$.

We will find the transition matrix from S to T and use it to find $[\mathbf{v}]_T$.

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -2 \\ 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{v}]_T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$$