6. Let $L : P_2 \to P_3$ be the linear transformation defined by $L(p(t)) = t^2 p'(t)$.
   (a) Find a basis for and the dimension of ker L.
   Let $at^2 + bt + c$ be in the kernel of L.

$$\begin{aligned} L(at^2+bt+c) &= 0 \\ t^2(2at+b) &= 0 \\ 2at^2+bt^2 &= 0 \end{aligned}$$

   Therefore $a = 0$ and $b = 0$, but c can be anything. So elements in $\ker L$ are polynomials of the form c. Therefore $\{1\}$ is a basis for $\ker L$ and hence the dimension of the kernel is 1.

   (b) Find a basis for and the dimension of range L.

$$L(at^2+bt+c) = t^2(2at+b) = 2at^3+bt^2$$

   Therefore $\{t^3, t^2\}$ is a basis for the rangel of L and hence the range has dimension 2.

19. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

   (a) Prove that L is invertible.
   To prove L is invertible we need to show L is one-to-one and onto. However since both vector spaces have dimension 3, then by Corollary 6.2 we only need to show one-to-one. We will do this by showing $\ker L = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.

   Let $\begin{bmatrix}a\\b\\c\end{bmatrix}$ be in $\ker L$.

$$\begin{aligned} L\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) &= 0 \\ a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix} \\ \begin{bmatrix}a+c\\2a+b+c\\3a+b\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $a = b = c = 0$ and hence $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the kernel if and only if $a = b = c = 0$.

Thus $\ker L = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Therefore L is one-to-one and hence L is invertible.

(b) Find $L^{-1}\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$.

We need to find where $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ came from under L.

$$\begin{aligned} L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) &= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} a + c \\ 2a + b + c \\ 3a + b \end{bmatrix} &= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$\text{Therefore } L\left(\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \text{ and hence } L^{-1}\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

22. Find a linear transformation $L : \mathbb{R}_2 \to \mathbb{R}_3$ such that $S = \{[1 \quad -1 \quad 2], [3 \quad 1 \quad -1]\}$ is a basis for range L.

    Let's send the basis vectors in $\mathbb{R}_2$ to the vectors given. So $L([1\ 0]) = [1\ -1\ 2]$ and $L([0\ 1] = [3\ 1\ -1]$. Then we can find the rule for L below.

    $$L([a\ b]) = a[1\ -1\ 2] + b[2\ 1\ -1] = [a + 2b \quad b - a \quad 2a - b]$$