10. Let $L : P_1 \to P_2$ be defined by $L(p(t)) = tp(t) + p(0)$. Consider the ordered bases $S = \{t, 1\}$ and $S' = \{t+1, t-1\}$ for P_1, and $T = \{t^2, t, 1\}$ and $T' = \{t^2+1, t-1, t+1\}$ for P_2. Find the representation of L with respect to

(a) S and T

$$L(t) = t(t) + 0 = t^2 \qquad [t^2]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$L(1) = t(1) + 1 = t \qquad [t]_T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) S' and T'

$$L(t+1) = t(t+1) + 1 = t^2 + t + 1$$

$$\begin{aligned} t^2 + t + 1 &= a(t^2+1) + b(t-1) + c(t+1) \\ t^2 + t + 1 &= at^2 + (b+c)t + (a-b+c) \\ a &= 1 \\ b + c &= 1 \\ 1 - b + c &= 1 \\ 2c &= 1 \\ c &= \frac{1}{2} \\ b &= \frac{1}{2} \end{aligned}$$

$$[t^2+t+1]_{T'} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$L(t-1) = t(t-1) - 1 = t^2 - t - 1$$

$$\begin{aligned} t^2 - t - 1 &= a(t^2+1) + b(t-1) + c(t+1) \\ t^2 - t - 1 &= at^2 + (b+c)t + (a-b+c) \\ a &= 1 \\ b + c &= -1 \\ 1 - b + c &= -1 \\ 2c &= -3 \\ c &= -\frac{3}{2} \\ b &= \frac{1}{2} \end{aligned}$$

$$[t^2 - t - 1]_{T'} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

(c) Find $L(-3t-3)$ by using the definition of L and the matrices obtained in parts (a) and (b).

Using the definition:

$$L(-3t-3) = t(-3t-3) - 3 = -3t^2 - 3t - 3$$

Using part (a):

$$[L(-3t-3)]_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$$

$$L(-3t-3) = -3t^2 - 3t - 3$$

Using part (b):

$$-3t - 3 = -3(t+1) + 0(t-1)$$

$$[-3t-3]_{S'} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$[L(-3t-3)]_{T'} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

$$L(-3t-3) = -3(t^2+1) - \frac{3}{2}(t-1) - \frac{3}{2}(t+1) = -3t^2 - 3t - 3$$