20. Let $A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) Find a basis for the eigenspace associated with the eigenvalue $\lambda_1 = 1$.

$$\begin{bmatrix} -1 & -2 & -3 & -4 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_4 &= 0 \\ x_2 + 3x_3 + 2(0) &= 0 \\ x_2 &= -3x_3 \\ x_1 + 2x_2 + 3x_3 + 4(0) &= 0 \\ x_1 &= -2x_2 - 3x_3 \\ x_1 &= 3x_3 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 3x_3 \\ -3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Therefore $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace associated with $\lambda = 1$.

(b) Find a basis for the eigenspace associated with the eigenvalue $\lambda_2 = 2$.

$$\begin{bmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_4 &= 0 \\ x_3 - 0 &= 0 \\ x_3 &= 0 \\ -2x_2 - 3(0) - 4(0) &= 0 \\ x_2 &= 0 \end{aligned}$$

Note that x_1 can be anything.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace associated with $\lambda = 1$.

22. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A^2 and verify exercise 21.

$$A^2 = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -1 & 8 \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} \lambda - 5 & 4 \\ 1 & \lambda - 8 \end{vmatrix} &= 0 \\ (\lambda - 5)(\lambda - 8) - 4 &= 0 \\ \lambda^2 - 13\lambda + 36 &= 0 \\ (\lambda - 9)(\lambda - 4) &= 0 \\ \lambda &= 4, 9 \end{aligned}$$

Note that the eigenvalues of A are 2 and -3. So the eigenvalues of A^2 are in fact the eigenvalues of A squared.