Name ______________________

Sec # ________

Math 35
Exam 1

February 27, 2009

- Please show all of your work clearly.
- Partial credit will be given where appropriate.
- Please check that you have all the problems.

Problem	Points	Possible
1		12
2		4
3		6
4		5
5		6
6		10
7		9
8		8
Score		60
Percent		100

1. **True or False.** Determine whether the following statements are true or false. If the statement is always true, give a brief justification. If the statement is sometimes false, give a counterexample or brief justification.

T **[F]** If A and B are $n \times n$ matrices, then $AB = BA$.

Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

T **[F]** If A and B are $n \times n$ matrices and $AB = 0$, then $A = 0$ or $B = 0$.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[T] **F** The inverse of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

This is true because when you multiply the two matrices you get I_3.

T **[F]** If f is a matrix transformation such that $f(\mathbf{u}) = A\mathbf{u}$ and $f : \mathbb{R}^3 \to \mathbb{R}^2$, then A is a 3×2 matrix.

Since $f : \mathbb{R}^3 \to \mathbb{R}^2$, then $\mathbf{u}$ must be 3×1 and $f(\mathbf{u})$ must be 2×1. So in order for $A\mathbf{u}$ to be defined and result in a 2×1 matrix, A must be 2×3.

2. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$. Compute $\mathbf{u} \cdot \mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = (-1)(3) + 4(-1) + 3(0) = \boxed{-7}$$

3. Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix}$ and let $f(\mathbf{u}) = A\mathbf{u}$. Is $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the range of f?

$$\begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x + 3y \\ -2x + 3y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$x + 3y = 2$$
$$-2x + 3y = 1$$

Subtracting the second equation from the first, you get $3x = 1$. So $x = \frac{1}{3}$. Substituting back in to the first equation, we get $\frac{1}{3} + 3y = 2$ and therefore $y = \frac{5}{9}$.

Since we found $\mathbf{u}$ such that $f(\mathbf{u}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we can conclude that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the range of f.

4. Find the determinant of $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$$\det A = (1)(1)(2)(2) - (1)(1)(2)(3) = 4 - 6 = \boxed{-2}$$

5. Consider the linear equation $AC^T\mathbf{x} = \mathbf{b}$ where

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix};\ C^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix};\ \text{and } \mathbf{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

Find the solution $\mathbf{x}$.

$$\begin{aligned} AC^T\mathbf{x} &= \mathbf{b} \\ C^T\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= (C^T)^{-1}A^{-1}\mathbf{b} \\ \mathbf{x} &= (C^{-1})^T A^{-1}\mathbf{b} \\ &= \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 13 \\ 19 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 45 \\ 84 \end{bmatrix}} \end{aligned}$$

6. (a) Find the reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 9 \\ 3 & 2 & 4 \end{bmatrix}$$

$$\begin{matrix} \\ 2R_1 + R_2 \to R_2 \\ 3R_1 - R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & -2 & -10 \end{bmatrix}$$

$$\begin{matrix} \\ \\ 2R_2 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Is A singular or nonsingular? Explain.

A is singular because its reduced row echelon form is not the identity matrix.

7. After completing several steps of Gaussian elimination to solve a system of linear equations, we have the augmented matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & a^2 - 9 & a - 3 \end{bmatrix}$$

(a) For what value(s) of a is the system inconsistent? Explain.

The system is inconsistent if $a^2 - 9 = 0$, but $a - 3 \neq 0$. Thus if $\boxed{a = -3}$, then the system is inconsistent.

(b) For what value(s) of a does the system have a unique solution? Explain.

As long as both $a^2 - 9$ and $a - 3$ are non-zero, the system will have a unique solution. Thus $\boxed{a \neq \pm 3}$.

(c) For what value(s) of a does the system have infinitely many solutions? Explain.

The system has infinitely many solutions if both $a^2 - 9$ and $a - 3$ equal zero. Thus there will be infinitely many solutions if $\boxed{a = 3}$.

8. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$.

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 - R_2 \to R_2 \\ R_1 - R_3 \to R_3 \end{array} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_2 - R_3 \to R_3 \end{array} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{r} 2R_1 + 3R_3 \to R_1 \\ 2R_2 + R_3 \to R_2 \end{array} \left[\begin{array}{cccccc} 2 & 4 & 0 & 2 & -3 & 3 \\ 0 & 2 & 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \left[\begin{array}{cccccc} 2 & 0 & 0 & -2 & 3 & 1 \\ 0 & 2 & 0 & 2 & -3 & 1 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \\ -\frac{1}{2}R_3 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & \frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$