1. **True or False.** Determine whether the following statements are true or false. If the statement is always true, give a brief justification. If the statement is sometimes false, give a counterexample or brief justification.

[T] **F** If A and B are $n \times n$ matrices, then $\det(AB^{-1}) = \dfrac{\det A}{\det B}$.

$$\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A) \cdot \frac{1}{\det B} = \frac{\det A}{\det B}$$

T **[F]** Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in a vector space V. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Consider the vector space P_1, then $\{1, t\}$ is a linearly independent set, but $\{1, t, t-1\}$ is a linearly dependent set.

[T] **F** If $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, then $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is in the span of S.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

[T] **F** If $P = (1,3)$, $Q = (4,2)$, $R = (-1,1)$, $S = (2,0)$, then $\overrightarrow{PQ} = \overrightarrow{RS}$.

$$\overrightarrow{PQ} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \overrightarrow{RS}$$

2. Find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 4 & 2 & 2 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 4 & 2 & 2 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix} &= -2 \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & -2 \end{vmatrix} \\ &= -2\Big[1(-2+1) - 3(-4+1) + 1(2-1)\Big] \\ &= -2(-1+9+1) \\ &= -2(9) \\ &= -18 \end{aligned}$$

3. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$.

(a) Compute the adjoint of A.

$$\begin{aligned}
A_{11} &= 3 - 2 = 1 \\
A_{12} &= -(6+1) = -7 \\
A_{13} &= 4 + 1 = 5 \\
A_{21} &= -(3-4) = 1 \\
A_{22} &= 3 + 2 = 5 \\
A_{23} &= -(2+1) = -3 \\
A_{31} &= 1 - 2 = -1 \\
A_{32} &= -(1-4) = 3 \\
A_{33} &= 1 - 2 = -1
\end{aligned}$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -7 & 5 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

(b) Find A^{-1}.

$$\det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix} = 1(3-2) - 1(6+1) + 2(4+1) = 1 - 7 + 10 = 4$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{7}{4} & \frac{5}{4} & \frac{3}{4} \\ \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

4. Consider the system of equations below. Use Cramer's rule to find x_2.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 2x^2 - 2x^3 &= -2 \\ x_1 + x_2 + 2x_3 &= -4 \end{aligned}$$

$$\det \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & -2 \\ 1 & 1 & 2 \end{bmatrix} = 2(4+2) - 1(6+2) + 1(3-2) = 12 - 8 + 1 = 5$$

$$A_2 = \det \begin{bmatrix} 2 & 6 & 1 \\ 3 & -2 & -2 \\ 1 & -4 & 2 \end{bmatrix} = 2(-4-8) - 6(6+2) + 1(-12+2) = -24 - 48 - 10 = -82$$

$$x_2 = \frac{-82}{5}$$

5. Let V be the set of all positive real numbers. Define $\mathbf{u} \oplus \mathbf{v} = uv$ and $c \odot \mathbf{v} = v^c$.

   (a) Is V closed with respect to $\oplus$? Justify your answer.
   If u and v are positive real numbers, then uv is a positive real number. So V is closed with respect to $\oplus$.

   (b) Show that vector space property 2(b) holds. (See back page.)
   Let c, d be real numbers and let u be a positive real number.

   $$(c+d) \odot u = u^{c+d} = u^c u^d = u^c \oplus u^d = (c \odot u) \oplus (d \odot u)$$

   Thus property $2(b)$ holds.

   (c) What is the zero in this vector space?

   $$u \oplus 1 = u(1) = u$$

   Thus 1 is the zero in V.

   (d) What is the negative of 5 in this vector space?

   $$\frac{1}{5} \oplus 5 = \left(\frac{1}{5}\right)(5) = 1$$

   Thus the negative of 5 is $\frac{1}{5}$.

6. Let W be the set of all vectors of the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$. Is W a subspace of $\mathbb{R}^3$? Justify your answer.

   Notice that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are in W, but $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, which is not in W. Thus W is not closed with respect to addition, so W is not a vector space.

7. Let A be a 4×4 matrix such that the row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Find a spanning set for the null space of A.

   From row 2 we see that $c + 3d = 0$ and hence $c = -3d$.

   From row 1 we see that $a + 2b - c + d = 0$ and hence $a = -2b - c + d = -2b + 4d$.

   Thus $\mathbf{x} = \begin{bmatrix} -2b+4d \\ b \\ -3d \\ d \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

   Therefore $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A.

8. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right\}$. Is S a linearly independent set? Justify your answer.

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & -2 & -1 & -3 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & -2 & -1 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

From this last matrix we can see that the system above will have non-trivial solutions, therefore the set is linearly dependent.

A **real vector space** is a set V with two operations $\oplus$ and $\odot$ such that the following properties hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are elements in V, then $\mathbf{u} \oplus \mathbf{v}$ is in V. (Closure with respect to $\oplus$)

   (a) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v}$ in V.

   (b) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.

   (c) There exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$ and $\mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

   (d) For each $\mathbf{u}$ in V there exists an element $-\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}$ and $-\mathbf{u} \oplus \mathbf{u} = \mathbf{0}$.

2. If $\mathbf{u}$ is any element in V and c is any real number, then $c \odot \mathbf{u}$ is in V. (Closure with respect to $\odot$)

   (a) $c \odot (\mathbf{u} \oplus \mathbf{v}) = (c \odot \mathbf{u}) \oplus (c \odot \mathbf{v})$ for $\mathbf{u}, \mathbf{v}$ in V and any real number c.

   (b) $(c + d) \odot \mathbf{u} = (c \odot \mathbf{u}) \oplus (d \odot \mathbf{u})$ for any $\mathbf{u}$ in V and any real numbers c and d.

   (c) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

   (d) $1 \odot \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.