

1. **True or False.** Determine whether the following statements are true or false. If the statement is always true, give a brief justification. If the statement is sometimes false, give a counterexample or brief justification.

T **F** The vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ are orthogonal.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 0$$

Therefore the two vectors are orthogonal.

T **F** If A is a 3×4 matrix then the rank of A is less than or equal to 3.

Since the rank of A is equal to the dimension of the row space and there are only three rows, the dimension must be less than or equal to 3.

T **F** If $W = \text{span} \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, then the dimension of W is 3.

The vectors w_1, w_2 and w_3 may not be linearly independent. For example $\{1, t, t - 1\}$ spans P_2 , but is not a basis for P_2 .

T **F** Let $L : P_1 \rightarrow \mathbb{R}$ be defined by $L(at + b) = a + b$, then $2t + 1$ is in the kernel of L .

$L(2t + 1) = 2 + 1 = 3 \neq 0$, so $2t + 1$ is not in the kernel of L .

T **F** Let $L : P_1 \rightarrow \mathbb{R}$ be defined by $L(at + b) = a + b$, then 2 is in the range of L .

$L(t + 1) = 1 + 1 = 2$, therefore 2 is in the range of L .

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

(a) Find a basis for the null space of A .

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & -2 \\ 0 & 2 & 8 & -4 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b + 4c - 2d = 0$$

$$b = 2d - 4c$$

$$a + 2b + 3c - d = 0$$

$$a = -2b - 3c + d$$

$$a = -2(2d - 4c) - 3c + d$$

$$a = 5c - 3d$$

$$\mathbf{x} = \begin{bmatrix} 5c - 3d \\ 2d - 4c \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore $\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space.

- (b) Determine the rank of
- A
- . Be sure to justify your answer.

Since there are 2 nonzero rows in the previous row reduction, the rank of A is 2.

- (c) Find a basis for the row space of
- A
- .

We can take the two non-zero rows from the row reduction. So $\{[1 \ 2 \ 3 \ -1], [0 \ 1 \ 4 \ -2]\}$ is a basis for the row space of A .

- (d) Are the columns of
- A
- linearly independent. Be sure to justify your answer.

A is a 4×4 matrix, but the rank of A is 2. Therefore the columns are linearly dependent.

3. Let
- $S = \{t, 1\}$
- and
- $T = \{t - 1, t + 1\}$
- , then
- S
- and
- T
- are bases for
- P_1
- .

- (a) If
- $[p(t)]_T = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$
- , what does
- $p(t)$
- equal?

$$p(t) = 2(t - 1) - 3(t + 1) = -t - 5$$

- (b) Find the transition matrix from the
- S
- basis to the
- T
- basis,
- $P_{T \leftarrow S}$
- .

$$t = a(t - 1) + b(t + 1)$$

$$1 = a(t - 1) + b(t + 1)$$

$$t = (a + b)t + (b - a)$$

$$1 = (a + b)t + (b - a)$$

$$a + b = 1$$

$$a + b = 0$$

$$b - a = 0$$

$$b - a = 1$$

$$2b = 1$$

$$2b = 1$$

$$b = \frac{1}{2}$$

$$b = \frac{1}{2}$$

$$a = \frac{1}{2}$$

$$a = -\frac{1}{2}$$

$$[t]_T = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$[1]_T = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$P_{T \leftarrow S} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- (c) Determine
- $[2t + 3]_T$
- .

$$[2t + 3]_T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} [2t + 3]_S = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$$

4. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ be a basis for W . Find an orthonormal basis for W .

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = -\frac{1}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{4}{3} \\ \frac{11}{6} \end{bmatrix} = 6 \begin{bmatrix} 5 \\ -8 \\ 11 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{5}{\sqrt{210}} \\ -\frac{8}{\sqrt{210}} \\ \frac{11}{\sqrt{210}} \end{bmatrix} \right\}$$

5. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $L \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a + b \\ a - b \\ b \end{bmatrix}$.

(a) Compute $L \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

$$L \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

(b) Is L a linear transformation? Justify your response.

$$L \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) + L \left(\begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} a + b \\ a - b \\ b \end{bmatrix} + \begin{bmatrix} c + d \\ c - d \\ d \end{bmatrix} = \begin{bmatrix} a + b + c + d \\ a - b + c - d \\ b + d \end{bmatrix}$$

$$L \left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right) = L \left(\begin{bmatrix} a + c \\ b + d \end{bmatrix} \right) = \begin{bmatrix} a + c + b + d \\ a + c - (b + d) \\ b + d \end{bmatrix} = \begin{bmatrix} a + b + c + d \\ a - b + c - d \\ b + d \end{bmatrix}$$

Thus L preserves addition.

$$cL\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = c\begin{bmatrix} a+b \\ a-b \\ b \end{bmatrix} = \begin{bmatrix} c(a+b) \\ c(a-b) \\ cb \end{bmatrix} = \begin{bmatrix} ca+cb \\ ca-cb \\ cb \end{bmatrix}$$

$$L\left(c\begin{bmatrix} a \\ b \end{bmatrix}\right) = L\left(\begin{bmatrix} ca \\ cb \end{bmatrix}\right) = \begin{bmatrix} ca+cb \\ ca-cb \\ cb \end{bmatrix}$$

Thus L preserves scalar multiplication.

Therefore L is a linear transformation.

6. Let $L : P_2 \rightarrow P_3$ be the linear transformation defined by $L(p(t)) = t^2 p'(t)$.

(a) Find a basis for the kernel of L .

Let $at^2 + bt + c$ be in P_2 .

$$L(at^2 + bt + c) = 0$$

$$t^2(2at + b) = 0$$

$$2at^3 + bt^2 = 0$$

Therefore $a = 0$, $b = 0$ and c can be anything. Thus $B = \{1\}$ is a basis for $\ker L$.

(b) What is the dimension of the range of L . Justify your response.

$$\dim(\ker L) + \dim(\text{range } L) = \dim P_2$$

$$1 + \dim(\text{range } L) = 3$$

Therefore $\dim(\text{range } L) = 2$.