7. Define \( \phi : \mathbb{Z} \to H \) by \( \phi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \).

We first prove one to one. Let \( n, m \in \mathbb{Z} \) such that \( \phi(n) = \phi(m) \). So \( \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \) which implies \( n = m \). Thus \( \phi \) is one to one.

We now prove onto. Let \( \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in H \), then \( n \in \mathbb{Z} \) and \( \phi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \). Thus \( \phi \) is onto.

Finally let’s check that \( \phi \) preserves the group structure.

\[
\phi(n + m) = \begin{bmatrix} 1 & n + m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \phi(n)\phi(m)
\]

Thus \( \phi \) is an isomorphism.

12. Let \( \phi : \mathbb{R} \to \mathbb{R}^+ \) given by \( \phi(x) = 10^x \). Prove \( \phi \) is an isomorphism.

**Proof.** Let \( a, b \in \mathbb{R} \) such that \( \phi(a) = \phi(b) \). Thus \( 10^a = 10^b \). Hence \( a = b \).

Let \( y \in \mathbb{R}^+ \). Then \( \log y \in \mathbb{R} \). Moreover \( \phi(\log y) = 10^{\log y} = y \). Thus \( \phi \) is onto.

Let \( a, b \in \mathbb{R} \). Then \( \phi(a + b) = 10^{a+b} = 10^a \cdot 10^b = \phi(a)\phi(b) \).

Thus \( \phi \) is an isomorphism. \( \square \)

14. Let \( \phi : \mathbb{C}^* \to \mathbb{C}^* \) defined by \( \phi(a + bi) = a - bi \). Prove that \( \phi \) is an automorphism.

**Proof.** Let’s first prove that \( \phi \) preserves the group structure. Let \( a + bi, c + di \in \mathbb{C} \). Then we have the following.

\[
\phi(a + bi)\phi(c + di) = (a - bi)(c - di)
\]

\[
= (ac - bd) - (ad + bc)i
\]

\[
= \phi\left((ac - bd) + (ad + bc)i\right)
\]

\[
= \phi\left((a + bi)(c + di)\right)
\]

Now let’s prove one-to-one. Suppose \( \phi(a + bi) = \phi(c + di) \). Thus \( a - bi = c - di \) and hence \( a = c \) and \( b = d \). Therefore \( a + bi = c + di \). Thus \( \phi \) is one-to-one.

Now we will prove onto. Let \( a + bi \in \mathbb{C} \). Then \( a - bi \in \mathbb{C} \) and \( \phi(a - bi) = a + bi \). Therefore \( \phi \) is onto.

Hence \( \phi \) is an automorphism on \( \mathbb{C} \). \( \square \)

16. Suppose \( (m, n) = 1 \) and let \( \phi : \mathbb{Z}_n \to \mathbb{Z}_n \) be defined by \( \phi([a]) = m[a] \). Prove or disprove that \( \phi \) is an automorphism.
Proof. We first need to check well-defined. Suppose \([a] = [b]\). Then \(a \equiv b \pmod{n}\) and hence \(ma \equiv mb \pmod{n}\). Thus \(m[a] = m[b]\). Therefore \(\phi\) is well-defined.

Let’s check that \(\phi\) preserves group structure.

\[
\phi([a] + [b]) = \phi([a + b]) = m[a + b] = m([a] + [b]) = m[a] + m[b] = \phi([a]) + \phi([b])
\]

Now let’s prove one-to-one. Suppose \(\phi([a]) = \phi([b])\). Then \(m[a] = m[b]\). So \(ma \equiv mb \pmod{n}\). Since \((m, n) = 1\) this implies that \(a \equiv b \pmod{n}\). Thus \([a] = [b]\) and hence \(\phi\) is one-to-one.

Now let’s prove onto. Let \([a] \in \mathbb{Z}_n\). Since \((m, n) = 1\) there are integers \(c\) and \(d\) such that \(1 = mc + nd\). Multiplying this by \(a\) gives \(a = mac + nad\) and hence \(a \equiv mac \pmod{n}\). Thus \([a] = [mac]\). Therefore we have the following: \(\phi([ac]) = m[ac] = [mac] = [a]\). Thus \(\phi\) is onto. Therefore \(\phi\) is an automorphism.

\[\square\]

18. For each \(a\) in the group \(G\), define a mapping \(t_a : G \to G\) by \(t_a(x) = axa^{-1}\). Prove that \(t_a\) is an automorphism.

Proof. We first prove one-to-one. Suppose \(x, y \in G\) such that \(t_a(x) = t_a(y)\). Thus we have the following.

\[
t_a(x) = t_a(y) \\
axa^{-1} = aya^{-1} \\
x = y
\]

Thus \(t_a\) is one-to-one.

We now prove onto. Let \(x \in G\). Then \(a^{-1}xa \in G\) and \(t_a(a^{-1}xa) = a(a^{-1}xa)a^{-1} = x\). Thus \(t_a\) is onto.

Finally we prove \(t_a\) preserves the group structure.

\[
t_a(xy) = axya^{-1} = axa^{-1}aya^{-1} = t_a(x)t_a(y)
\]

Thus \(t_a\) is an isomorphism. \[\square\]

28. Suppose that \(G\) and \(H\) are isomorphic groups. Prove that \(G\) is abelian if and only if \(H\) is abelian.

Proof. Assume \(G\) and \(H\) are isomorphic, then there exists an isomorphism \(\phi : G \to H\).

Assume \(G\) is abelian. Let \(x, y \in H\). Since \(\phi\) is onto there exists \(a, b \in G\) such that \(\phi(a) = x\) and \(\phi(b) = y\). Then we have \(xy = \phi(a)\phi(b) = \phi(ab)\) since \(\phi\) preserves group structure. Since \(G\) is abelian, \(\phi(ab) = \phi(ba)\). Finally, since \(\phi\) preserves group structure \(\phi(ba) = \phi(b)\phi(a) = yx\). Thus we have shown that \(xy = yx\). Hence \(H\) is abelian.

Assume \(H\) is abelian. Let \(a, b \in G\). Then \(\phi(a), \phi(b) \in H\) and since \(H\) is abelian \(\phi(a)\phi(b) = \phi(b)\phi(a)\). Thus we have \(\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba)\). Now, since \(\phi\) is one-to-one and \(\phi(ab) = \phi(ba)\), then we have that \(ab = ba\). Thus \(G\) is abelian. \[\square\]

33. If \(G\) and \(G'\) are groups and \(\phi : G \to G'\) is an isomorphism, prove that \(a\) and \(\phi(a)\) have the same order for any \(a \in G\).
Proof. Let \(|a| = m\), so \(a^m = e\). Then \(\phi(a)^m = \phi(a^m) = \phi(e) = e'\). Where \(e'\) is the identity element of \(G'\).

Suppose there is some positive integer \(j\) such that \(\phi(a)^j = e'\). Then \(e' = \phi(a)^j = \phi(a^j)\), but \(\phi(e) = e'\). Since \(\phi\) is one-to-one, \(a^j = e\). However, \(|a| = m\), hence \(m \leq j\).

Therefore we have shown that \(\phi(a)^m = e'\) and \(m\) is the smallest positive such power. Hence \(|\phi(a)| = m\).

34. Suppose that \(\phi\) is an isomorphism from the group \(G\) to the group \(G'\).

(a) Prove that if \(H\) is a subgroup of \(G\), then \(\phi(H)\) is a subgroup of \(G'\).

\[ \text{Proof.} \] Since \(H\) is a subgroup of \(G\), \(e \in H\). Moreover \(\phi(e) = e'\). Thus \(e' \in \phi(H)\).

Let \(x, y \in \phi(H)\). So there exists \(a, b \in H\) such that \(\phi(a) = x\) and \(\phi(b) = y\). Thus \(xy = \phi(a)\phi(b) = \phi(ab)\). Since \(H\) is a group \(ab \in H\). Thus \(xy = \phi(ab) \in \phi(H)\).

Let \(z \in \phi(H)\). So there exists \(c \in H\) such that \(\phi(c) = z\). Thus \(z^{-1} = \phi(c)^{-1} = \phi(c^{-1})\).

Since \(H\) is a group \(a^{-1} \in H\). Thus \(z^{-1} = \phi(c^{-1}) \in \phi(H)\).

Therefore by the subgroup test, \(\phi(H)\) is a subgroup of \(G'\).

(b) Suppose that \(\phi\) is an isomorphism from \(G\) to \(G'\). Prove that if \(K\) is a subgroup of \(G'\), then \(\phi^{-1}(K)\) is a subgroup of \(G\).

\[ \text{Proof.} \] Since \(\phi(e) = e' \in K\), \(e \in \phi^{-1}(K)\). Thus \(\phi^{-1}(K) \neq \emptyset\).

Let \(a, b \in \phi^{-1}(K)\). So \(\phi(a), \phi(b) \in K\). Since \(K\) is a subgroup of \(G'\), \(\phi(a)\phi(b) \in K\). Therefore we have the following.

\[ \phi(ab) = \phi(a)\phi(b) \in K \]

Thus \(ab \in \phi^{-1}(K)\). Hence \(\phi^{-1}(K)\) is closed.

Let \(a \in \phi^{-1}(K)\). So \(\phi(a) \in K\). Since \(K\) is a subgroup of \(G'\), \(\phi(a)^{-1} \in K\). Therefore \(\phi(a^{-1}) = \phi(a)^{-1} \in K\). Thus \(a^{-1} \in \phi^{-1}(K)\).

Therefore \(\phi^{-1}(K)\) is a subgroup of \(G'\).