2. Show that $H = \{I, -I\}$ is a normal subgroup of $GL_2(\mathbb{R})$, where $I$ is the 2 by 2 identity matrix.

Proof. The inverse of $I$ is $I$ and the inverse of $-I$ is $-I$. Thus $H$ contains inverses. $I \cdot I = I \in H$, $I \cdot (-I) = -I \in H$, and $(-I)(-I) = I \in H$. Thus $H$ is closed. Therefore since $H$ is nonempty, closed and contain inverse we have that $H$ is a subgroup of $GL_2(\mathbb{R})$.

Now let’s prove that $H$ is a normal subgroup of $GL_2(\mathbb{R})$. Let $A \in GL_2(\mathbb{R})$, then $AIA^{-1} = AA^{-1} = I \in H$ and $A(-I)A^{-1} = -AA^{-1} = -I \in H$. Thus by the normal subgroup test $H$ is normal. \[\square\]

16. If $H \leq G$ and $K \triangleleft G$, prove that $HK = KH$.

Proof. Let $H \leq G$, $K \triangleleft G$. Let $hk \in HK$. Since $K \triangleleft G$, $hkh^{-1} \in K$. Thus $hkh^{-1} = k_1$ for some $k_1 \in K$. Therefore $hk = k_1h \in KH$. Thus $HK \subseteq KH$.

Let $kh \in KH$. Then $h^{-1}kh = h^{-1}k(h^{-1})^{-1} \in K$, so $h^{-1}kh = k_1$ for some $k_1 \in K$. Thus $kh = hkh_k \in HK$. Therefore $KH \subseteq HK$.

Therefore $HK = KH$. \[\square\]

17. If $H \leq G$ and $K \triangleleft G$, prove that $HK \leq G$.

Proof. Since $H$ and $K$ are subgroups, $e \in H$ and $e \in K$, thus $e = ee \in HK$. Thus $HK \neq \emptyset$.

Let $h_1k_1, h_2k_2 \in HK$. Since $K \triangleleft G$, $Kk_2 = h_2K$. Therefore $h_1k_1h_2k_2 = h_1h_2k_3k_2 \in HK$, since $H$ and $K$ are closed. Therefore $HK$ is closed.

Let $hk \in HK$. Since $K \triangleleft G$, $hk = kh$. So $hk = k'h$ for some $k' \in K$. Therefore we have $(hk)^{-1} = (k'h)^{-1} = h^{-1}k'^{-1} \in HK$, since $H$ and $K$ contain inverses. Thus $HK$ contains inverses.

Therefore $HK \leq G$. \[\square\]

18. If $H \leq G$ and $K \triangleleft G$, prove that $H \cap K \triangleleft H$.

Proof. We have proven previously that $H \cap K \leq H$.

Let $h \in H$ and $x \in H \cap K$. Since $K \triangleleft G$ and $h \in G$ and $x \in K$, we know $h_2h^{-1} \in K$. Since $x \in H \cap K$, we know $x \in H$. Since $x \in H$ and $h \in H$, then $hkh^{-1} \in H$ since $H$ is closed. Thus $h_2h^{-1} \in K$ and $h_2h^{-1} \in H$, thus $hkh^{-1} \in H \cap K$.

Therefore $H \cap K \triangleleft H$. \[\square\]

21. Prove that if $H$ and $K$ are normal subgroups of $G$ such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$, $k \in K$.

Proof. Let $h \in H$, $k \in K$. Consider the element $hk^{-1}k^{-1}$. Since $H$ is normal, $hk^{-1}k^{-1} \in H$. Thus since $h \in H$, then by closure $hk^{-1}h^{-1} \in H$. Similarly since $K$ is normal, $hk^{-1} \in K$. Thus since $k^{-1} \in K$, then by closure $hk^{-1}k^{-1} \in K$. Therefore $hk^{-1}k^{-1} \in H \cap K$, but $H \cap K = \{e\}$. Thus $hk^{-1}k^{-1} = e$. Multiplying on the right by $k$, then $h$, we get $hk = kh$. \[\square\]

22. Prove that the center is a normal subgroup.
Proof. It was proven previously that $Z(G) \leq G$. Let $z \in Z(G)$ and $g \in G$. Then $zg = gz$, which implies $z = gzg^{-1}$. Thus, since $z \in Z(G)$, $gzg^{-1} \in Z(G)$. Therefore $Z(G) \triangleleft G$.  

32. Let $H$ be a subgroup of $G$ with index 2.

(a) Prove that $H$ is a normal subgroup of $G$.

Proof. Assume $[G : H] = 2$. Thus the set of left cosets equals $\{H, gH\}$ for some $g \in G$. Since $gH \neq H$, then $g \notin H$. Moreover $\{H, gH\}$ partitions $G$. Now since there are two left cosets there are two right cosets. Since $g \notin H$, $H$ and $Hg$ are distinct. Thus the set of right cosets are $\{H, Hg\}$ and is a partition of $G$.

Since $\{H, gH\}$ partitions $G$, $G - H = gH$. However, since $\{H, Hg\}$ also partitions $G$, $G - H = Hg$. Thus $gH = Hg$. Hence $H$ is a normal subgroup of $G$.  

(b) Prove that $g^2 \in H$ for all $g \in G$.

Proof. Let $g \in G$. By above we know that $H$ is a normal subgroup of $G$. Thus $G/H$ is a group. Since $[G : H] = 2$, then the order of $G/H$ is 2. Therefore any element in $G/H$ raised to the second power will be the identity in $G/H$. In other words $(gH)^2 = H$. However, $(gH)^2 = (gH)(gH) = g^2H$. Since $g^2H = H$, then $g^2 \in H$.  