1(a) Prove the set of all real numbers of the form \( m + n\sqrt{2} \), with \( m, n \in \mathbb{Z} \) is a subring of the real numbers.

**Proof.** Let \( S = \{ m + n\sqrt{2} | m, n \in \mathbb{Z} \} \). We want to prove that \( S \) is a subring of \( \mathbb{R} \).

Since \( \sqrt{2} \in S \), \( S \neq \emptyset \).

Let \( a + b\sqrt{2}, c + d\sqrt{2} \in S \). Then we have the following.

\[
(a + b\sqrt{2}) - (c + d\sqrt{2}) = (a - c) + (b - d)\sqrt{2} \in S
\]

\[
(a + b\sqrt{2})(c + d\sqrt{2}) = ac + 2bd + (ad + bc)\sqrt{2} \in S
\]

Thus, by the subring test, \( S \) is a subring of \( \mathbb{R} \).

16. Suppose that \( G \) is an abelian group with respect to addition, with identity element 0. Define a multiplication in \( G \) by \( ab = 0 \) for all \( a, b \in G \). Show that \( G \) forms a ring with respect to these operations.

**Proof.** Assume \( G \) is an abelian group and that \( ab = 0 \) for all \( a, b \in G \). Since \( ab = 0 \in G \), \( G \) is closed with respect to multiplication. Therefore we need only check the associative and distributive properties.

Let \( a, b, c \in G \). Then \( a(bc) = a0 = 0 \) and \( (ab)c = 0c = 0 \). Thus \( a(bc) = (ab)c \), so multiplication is associative.

Also \( a(b + c) = 0 \) since any product is 0. Also \( ab + ac = 0 + 0 = 0 \). Thus \( a(b + c) = ab + ac \), so the left distributive property holds. We can prove similarly the right distributive property holds. Thus \( G \) is a ring.

23. Prove that if \( a \) is a unit in a ring \( R \) with unity, then \( a \) is not a zero divisor in \( R \).

**Proof.** Let \( a \in R \) be a unit. Let \( b \in R \) such that \( ab = 0 \). Then we have the following.

\[
ab = 0
\]
\[
a^{-1}ab = a^{-1}(0)
\]
\[
b = 0
\]

Since \( b \) must be zero, \( a \) is not a zero divisor.

27. For a fixed element \( a \) of a ring \( R \), prove that the set \( \{ x \in R | ax = 0 \} \) is a subring of \( R \).

**Proof.** Let \( S = \{ x \in R | ax = 0 \} \). Since \( a(0) = 0, 0 \in S \). Thus \( S \neq \emptyset \).

Let \( x, y \in S \). Then \( a(x - y) = ax - ay = 0 - 0 = 0 \). Thus \( x - y \in S \).

Also \( a(xy) = (ax)y = 0(y) = 0 \), so \( xy \in S \).

Therefore \( S \) is a subring of \( R \).

28. For a fixed element \( a \) of a ring \( R \), prove that the set \( \{ xa | x \in R \} \) is a subring of \( R \).
50. Suppose \( R = \{xa|x \in R\} \). Since \( 0 = 0a \), then \( 0 \in S \). Therefore \( S \neq \emptyset \).

Let \( r, s \in S \). Then \( r = xa \) and \( s = ya \) for some \( x, y \in R \). Then \( r - s = xa - ya = (x - y)a \). Thus \( r - s \in S \).

Also, \( rs = xaya = (xay)a \). Thus \( rs \in S \).

Therefore \( S \) is a subring of \( R \).

37. Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

\begin{proof}
Let \( S = \{a \in R|a^2 = a\} \). Let \( x, y \in S \). So \( x^2 = x \) and \( y^2 = y \). Recall that \( R \) is a commutative ring, so we have the following.

\[
(xy)^2 = xyy = x^2y^2 = xy
\]

Thus \( xy \in S \).
\end{proof}

46. An element \( a \) of a ring \( R \) is called nilpotent if \( a^n = 0 \) for some positive integer \( n \). Prove that the set of nilpotent elements in a commutative ring \( R \) forms a subring of \( R \).

\begin{proof}
Let \( S = \{a \in R|a^n = 0 \text{ for some } n \in \mathbb{N}\} \).

Since \( 0^1 = 0 \), \( 0 \in S \). Thus \( S \neq \emptyset \).

Let \( a, b \in S \). Thus \( a^n = 0 \) and \( b^m = 0 \) for some \( n, m \in \mathbb{N} \). WLOG, \( n \geq m \). Therefore \((ab)^n = a^n b^n \) since \( R \) is commutative. Thus we have \((ab)^n = a^n b^n = 0(b^n) = 0 \). Hence \( ab \in S \).

Let \( r \in \mathbb{Z} \) such that \( r \geq n \), then \( a^{r-n} \in R \) and \( a^r = a^n a^{r-n} = 0(a^{r-n}) = 0 \). Similarly, let \( s \in \mathbb{Z} \) such that \( s \geq m \), then \( b^{m-s} \in R \) and \( b^s = b^m b^{s-m} = 0(b^{s-m}) = 0 \). Therefore we have the following.

\[
(a - b)^{n+m} = a^{n+m} + c_1 a^{n+m-1}b + c_2 a^{n+m-2}b^2 + \cdots + c_{m-1} a^{n+1}b^{m-1} + c_m a^n b^m + c_{m+1}a^{n-1}b^{m+1} + \cdots + c_{n+m-1}ab^{n+m-1} + b^{n+m}
\]

\[
= 0 + c_1(0)(b) + c_2(0)(b^2) + \cdots + c_{m-1}(0)b^{m-1} + c_m(0)(0) + c_{m-1}a^{n-1}(0) + \cdots + c_{n+m-1}(0) + 0
\]

\[
= 0
\]

Thus \( (a - b)^{n+m} = 0 \) and hence \( a - b \in S \).

Therefore \( S \) is a subring of \( R \).
\end{proof}

50. Suppose \( R \) is a ring in which all elements \( x \) satisfy \( x^2 = x \). (Such a ring is called a Boolean ring.)

(a) Prove that \( x = -x \) for each \( x \in R \).

\begin{proof}
Let \( x \in R \). Then we have the following.

\[
(x + x)^2 = x + x
\]

\[
x^2 + x^2 + x^2 + x^2 = x^2 + x^2
\]

\[
x^2 + x^2 = 0
\]

\[
x + x = 0
\]

\[
x = -x
\]
\end{proof}
(b) Prove that $R$ is commutative.

Proof. Let $x, y \in R$. Then we have the following.

\[
\begin{align*}
(x + y)^2 &= x + y \\
x^2 + xy + yx + y^2 &= x + y \\
x^2 + xy + yx + y^2 &= x^2 + y^2 \\
xy + yx &= 0 \\
xy &= -yx \\
xy &= yx \quad \text{(by part a)}
\end{align*}
\]