5. Suppose that \( \theta \) is an epimorphism from \( R \) to \( R' \) and that \( R \) has a unity. Prove that if \( a^{-1} \) exists for \( a \in R \), then \([\theta(a)]^{-1}\) exists and \([\theta(a)]^{-1} = \theta(a^{-1})\).

**Proof.** Let \( a \in R \) such that \( a^{-1} \) exists. Then we have the following.

\[
\theta(a)\theta(a^{-1}) = \theta(aa^{-1}) = \theta(e)
\]

Similarly \( \theta(a^{-1})\theta(a) = \theta(e) \). (Note that by problem 2, \( \theta(e) \) is the unity in \( R' \).) Thus \( \theta(a^{-1}) \) is the multiplicative inverse of \( \theta(a) \). In other words, \( \left( \theta(a) \right)^{-1} = \theta(a^{-1}) \).

6(b) Let \( S = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\} \). Define \( \theta : S \to \mathbb{Z} \) by \( \theta \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = z \). Describe \( \text{ker} \theta \) and prove \( S/\text{ker} \theta \) is isomorphic to \( \mathbb{Z} \).

**Proof.**

\[
\text{ker} \theta = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid \theta \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = 0 \right\}
\]

\[
= \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid z = 0 \right\}
\]

Thus the kernel is the set all matrices in \( S \) where the bottom right entry is 0.

Let \( K \) be the kernel of \( \theta \). Define \( \phi : S/K \to \mathbb{Z} \) by \( \phi \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + K \right) = z \).

Let’s check well-defined. So suppose \( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + K = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + K \). So \( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} - \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in K \).

In other words, \( \begin{pmatrix} x-a & y-b \\ 0 & z-c \end{pmatrix} \in K \). Thus \( z-c = 0 \), so \( z = c \). Therefore we have

\[
\phi \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = z = c = \phi \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right).
\]

Thus \( \phi \) is well-defined.

I will leave one-to-one, onto, homomorphism to you.

12. Let \( S = \{ m + n\sqrt{2} | m, n \in \mathbb{Z} \} \). Prove that \( \theta : S \to S \) defined by \( \theta(m + n\sqrt{2}) = m - n\sqrt{2} \) is an isomorphism.

**Proof.** Let \( a + b\sqrt{2}, c + d\sqrt{2} \in S \).

\[
\theta(a + b\sqrt{2} + c + d\sqrt{2}) = \theta(a + c + (b + d)\sqrt{2})
\]

\[
= a + c - (b + d)\sqrt{2}
\]

\[
= a - b\sqrt{2} + c - d\sqrt{2}
\]

\[
= \theta(a + b\sqrt{2}) + \theta(c + d\sqrt{2})
\]
\[ \theta((a + b\sqrt{2})(c + d\sqrt{2})) = \theta(ac + 2bd + (ad + bc)\sqrt{2}) = ac + 2bd - (ad + bc)\sqrt{2} = (a - b\sqrt{2})(c - d\sqrt{2}) = \theta(a + b\sqrt{2})\theta(c + d\sqrt{2}) \]

Thus \( \theta \) is a homomorphism.

Let \( x + y\sqrt{2} \in S \). Then \( \theta(x - y\sqrt{2}) = \theta(x + (-y)\sqrt{2}) = x - (-y)\sqrt{2} = x + y\sqrt{2} \). Thus \( \theta \) is onto.

Suppose \( \theta(r + s\sqrt{2}) = \theta(w + z\sqrt{2}) \) for some \( r, s, w, z \in \mathbb{Z} \). Then \( r - w = (s - z)\sqrt{2} \). If \( s - z \neq 0 \), then \( (s - z)\sqrt{2} \) is irrational, which is a contradiction. Thus \( s - z \) must be 0. So \( s = z \) and so \( r = z \). Therefore \( \theta \) is one-to-one.

Hence \( \theta \) is an isomorphism.

**Proof.**

Let \( \mathbb{R} = \left\{ \begin{bmatrix} m & 2n \\ n & m \end{bmatrix} \mid m, n \in \mathbb{Z} \right\} \) and \( \mathbb{R}' = \{ m + n\sqrt{2} \mid m, n \in \mathbb{Z} \} \). Prove that \( \mathbb{R} \) and \( \mathbb{R}' \) are isomorphic.

**Proof.** Define \( \theta : \mathbb{R} \to \mathbb{R}' \) by \( \theta \left( \begin{bmatrix} m & 2n \\ n & m \end{bmatrix} \right) = m + n\sqrt{2} \).

I’ll leave it to you to show that \( \theta \) is an isomorphism.

18(b) Suppose \( \theta : \mathbb{R} \to \mathbb{R}' \) is a homomorphism. Prove that if \( x \in \mathbb{R} \) is nilpotent, then \( \theta(x) \) is nilpotent in \( \mathbb{R}' \).

**Proof.** Let \( x \in \mathbb{R} \) be nilpotent. Then \( x^n = 0 \) for some \( n \in \mathbb{N} \). Then we have the following:

\[ \theta(x)^n = \theta(x^n) = \theta(0) = 0 \]

Thus \( \theta(x) \) is nilpotent.

25. Assume that \( \theta \) is an epimorphism from \( \mathbb{R} \) to \( \mathbb{R}' \). Prove the following.

(a) If \( I \) is an ideal of \( \mathbb{R} \), then \( \theta(I) \) is an ideal of \( \mathbb{R}' \).

(b) If \( I' \) is an ideal of \( \mathbb{R}' \), then \( \theta^{-1}(I') \) is an ideal of \( \mathbb{R} \).

**Proof.** Let \( \theta : \mathbb{R} \to \mathbb{R}' \) be an epimorphism and let \( I \) be an ideal in \( \mathbb{R} \) and \( I' \) be an ideal in \( \mathbb{R}' \).

(a) Since \( 0 \in I, \theta(0) \in \theta(I) \). Thus \( \theta(I) \neq \emptyset \).

Let \( a', b' \in \theta(I) \). Since \( \theta \) is onto, there exists \( a, b \in \mathbb{R} \) such that \( \theta(a) = a' \) and \( \theta(b) = b' \). Then we have the following.

\[ a' - b' = \theta(a) - \theta(b) = \theta(a - b) \in \theta(I) \]

Let \( r' \in \mathbb{R}' \). Since \( \theta \) is onto there exists \( r \in \mathbb{R} \) such that \( \theta(r) = r' \). Then \( r'a' = \theta(r)\theta(a) = \theta(ra) \in \theta(I) \). Similarly \( a'r' \in \theta(I) \).

Thus \( \theta(I) \) is an ideal in \( \mathbb{R}' \).

(b) Since \( \theta(0_R) = 0_{R'}, \theta_0 \in \theta^{-1}(I') \). Thus \( \theta^{-1}(I') \neq \emptyset \).

Let \( a, b \in \theta^{-1}(I') \). So \( \theta(a), \theta(b) \in I' \), thus \( \theta(a - b) = \theta(a) - \theta(b) \in I' \). Therefore \( a - b \in \theta^{-1}(I') \).

Let \( r \in \mathbb{R} \). Then \( \theta(ra) = \theta(r)\theta(a) \in I' \) since \( \theta(a) \in I' \). Therefore \( ra \in \theta^{-1}(I') \).

Thus \( \theta^{-1}(I') \) is an ideal in \( \mathbb{R} \).