9. (a) Let $F$ be a field. Prove that $I = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in F, a_0 + a_1 + \cdots + a_n = 0\}$ is an ideal in $F[x]$.

NOTE: You could prove that $I$ is an ideal directly, but here is another way to prove $I$ is an ideal.

**Proof.** Let $\phi_1 : F[x] \to F$ be the evaluation homomorphism at 1. Thus $\phi$ is a homomorphism.

We claim that the kernel of $\phi$ is $I$.

Let $f(x) \in I$ where $f(x) = a_0 + a_1 x + \cdots + a_k x^k$. Then $\phi(f(x)) = a_0 + a_1 + \cdots + a_k = 0$.

Therefore $f(x) \in \ker \phi$. Hence $I \subseteq \ker \phi$.

Let $g(x) \in \ker \phi$ where $g(x) = b_0 + b_1 x + \cdots + b_l x^l$. Then $0 = \phi(g(x)) = b_0 + b_1 + \cdots + b_l$.

Thus $g(x) \in I$. So $\ker \phi \subseteq I$.

Therefore $I$ is the kernel of a homomorphism and hence must be an ideal. \(\square\)

(b) Prove or disprove that $I$ is a principal ideal.

**Proof.** We claim that $I = (x - 1)$. Let $f(x) \in I$, where $f(x) = a_0 + a_1 x + \cdots + a_k x^k$.

Then we have the following.

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$

$$= (x - 1)[a_k x^{k-1} + (a_k + a_{k-1}) x^{k-2} + \cdots \cdots + (a_k + a_{k-1} + \cdots + a_2) x + (a_k + a_{k-1} + \cdots + a_2 + a_1)]$$

Thus $f(x) \in (x - 1)$. So $I \subseteq (x - 1)$.

Let $g(x) = b_0 + b_1 x + \cdots + b_l x^l$ where $g(x) \in (x - 1)$. Then $b_0 + b_1 + \cdots + b_l = g(1) = (1 - 1)h(1) = 0 \cdot h(1) = 0$. Thus $g(x) \in I$. So $(x - 1) \subseteq I$.

Therefore $I = (x - 1)$. \(\square\)

14. Prove or disprove that $R[x]$ is a field if $R$ is a field.

**Proof.** Suppose $x \in R[x]$ has a multiplicative inverse in $R[x]$. Then there exists $f(x) \in R[x]$ such that $x f(x) = e$. Since $R$ is a field, $R$ is an integral domain. Hence $R[x]$ is an integral domain. Thus $\deg(x f(x)) = \deg x + \deg f(x)$. Therefore $\deg(x f(x)) \geq 1$. However, $\deg(e) = 0$. Thus $x f(x) \neq e$. Hence $x$ does not have a multiplicative inverse, and thus $R[x]$ is not a field. \(\square\)

18. Let $R$ be a commutative ring with unity, and let $I$ be the principal ideal $I = (x)$ in $R[x]$. Prove that $R[x]/I$ is isomorphic to $R$.

**Proof.** Define $\phi : R[x] \to R$ by $\phi(f(x)) = f(0)$. This is just the evaluation homomorphism, hence it is a homomorphism.

Let $c \in R$, then $c \in R[x]$ and $\phi(c) = c$. Thus $\phi$ is onto.

Let $f(x) \in I$. Then $f(x) = x q(x)$ for some $q(x) \in R[x]$. Thus $f(0) = (0)q(0) = 0$, so $f(x) \in \ker \phi$.

Let $f(x) \in \ker \phi$. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $0 = f(0) = a_0$. Thus $f(x) = a_1 x + \cdots + a_n x^n = x(a_1 + \cdots + a_n x^{n-1})$. Hence $f(x) \in (x)$.

Therefore we have shown that $\phi$ is an epimorphism with kernel $(x)$. Therefore $R[x]/(x)$ is isomorphic to $R$. \(\square\)
22. Let $\theta : R \to S$ be an epimorphism. Define $\phi : R[x] \to S[x]$ by $\phi(a_0 + a_1 x + \cdots + a_n x^n) = \theta(a_0) + \theta(a_1) x + \cdots + \theta(a_n) x^n$. Prove that $\phi$ is an epimorphism.

Proof. Let $\theta : R \to S$ be an epimorphism. Let $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $g = \sum_{i=0}^{m} b_i x^i \in R[x]$.

WLOG $n \geq m$, so $g = \sum_{i=0}^{n} b_i x^i$ where $b_i = 0$ for all $i > m$.

\[
\phi(f + g) = \phi \left( \sum_{i=0}^{n} (a_i + b_i) x^i \right) \\
= \sum_{i=0}^{n} \theta(a_i + b_i) x^i \\
= \sum_{i=0}^{n} (\theta(a_i) + \theta(b_i)) x^i \\
= \sum_{i=0}^{n} \theta(a_i) x^i + \sum_{i=0}^{n} \theta(b_i) x^i \\
= \phi(f) + \phi(g)
\]

\[
\phi(fg) = \phi \left( \sum_{j+k=i}^{n+m} a_j b_k x^i \right) \\
= \sum_{j+k=i}^{n+m} \theta(a_j b_k) x^i \\
= \sum_{j+k=i}^{n+m} \theta(a_j) \theta(b_k) x^i \\
= \phi(f) \phi(g)
\]

Let $h = c_0 + c_1 x + \cdots + c_q x^q \in S[x]$. Since $\theta$ is an epimorphism, there exists $d_0, d_1, \ldots, d_q$ such that $\theta(c_i) = d_i$. Let $r = d_0 + d_1 x + \cdots + d_q x^q$, then $\phi(r) = h$. Thus $\phi$ is onto. Therefore $\phi$ is an epimorphism.

23. Describe the kernel of $\phi$ in problem 15.

Let $f = a_0 + a_1 x + \cdots + a_n x^n$.

\[
\ker \phi = \{ f \in R[x] | \phi(f) = 0 \} \\
= \{ f \in R[x] | \theta(a_i) = 0 \text{ for all } i \} \\
= \{ f \in R[x] | a_i \in \ker \theta \text{ for all } i \}
\]