5. Let $F$ be a field and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$.

(a) Prove that $x - 1$ is a factor of $f(x)$ if and only if $a_0 + a_1 + \cdots + a_n = 0$.

*Proof.* Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$.

$x - 1$ is a factor of $f(x)$ $\iff$ $f(1) = 0$

$\iff a_0 + a_1(1) + \cdots + a_n(1)^n = 0$

$\iff a_0 + a_1 + \cdots + a_n = 0$

\[ \blacksquare \]

(b) Prove that $x + 1$ is a factor of $f(x)$ if and only if $a_0 - a_1 + \cdots + (-1)^n a_n = 0$.

*Proof.* Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$.

$x + 1$ is a factor of $f(x)$ $\iff$ $f(-1) = 0$

$\iff a_0 + a_1(-1) + \cdots + a_n(-1)^n = 0$

$\iff a_0 - a_1 + \cdots + (-1)^n a_n = 0$

\[ \blacksquare \]

9. Let $F$ be a field. Prove that if $c$ is a zero of $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$, then $c^{-1}$ is a zero of $a_n + a_{n-1} x + \cdots + a_0 x^n$.

*Proof.* (Note: In order for $c^{-1}$ to exist $c \neq 0$.) Assume $c \in F$ is a zero of $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$. So $f(c) = 0$. Thus we have the following.

$a_0 + a_1 c + \cdots + a_n c^n = 0$

$(c^{-1})^n (a_0 + a_1 c + \cdots + a_n c^n) = 0$

$a_0 (c^{-1})^n + a_1 (c^{-1})^{n-1} + \cdots + a_n = 0$

Thus $c^{-1}$ is a zero of $a_0 x^n + a_1 x^{n-1} + \cdots + a_n$.

\[ \blacksquare \]

10. Let $f(x)$ and $g(x)$ be two polynomials over the field $F$, both of degree $n$ or less. Prove that if $m > n$ and if there exists $m$ distinct elements $c_1, c_2, \ldots, c_m$ of $F$ such that $f(c_i) = g(c_i)$ for $i = 1, 2, \ldots, m$, then $f(x) = g(x)$.

*Proof.* Let $\deg f(x), \deg g(x) \leq n$. Let $h(x) = f(x) - g(x)$. Suppose $h(x) \neq 0$, then $\deg h(x) \leq n$. Let $m > n$ and let $c_1, c_2, \ldots, c_m$ be elements in $F$ such that $f(c_i) = g(c_i)$. Thus $h(c_i) = 0$. In other words, $c_1, \ldots, c_m$ are zeros of $h$. Hence $h(x)$ has $m$ zeros. However $m > \deg h(x)$, which contradicts corollary 8.18. Thus $h(x)$ must equal 0 and hence $f(x) = g(x)$.

\[ \blacksquare \]

16. Let $f(x)$ be a polynomial of positive degree $n$ over the field $F$, and assume that $f(x) = (x - c)q(x)$ for some $c \in F$ and $q(x)$ in $F[x]$.

(a) Prove that $c$ and the zeros of $q(x)$ in $F$ are zeros of $f(x)$.

*Proof.* Clearly $c$ is a zero of $f(x)$ by the Factor Theorem.

Let $a$ be a zero of $q(x)$. Then $f(a) = (a - c)(q(a)) = (a - c)(0) = 0$. Thus $a$ is a zero of $f$.

\[ \blacksquare \]
(b) Prove that \( f(x) \) has no other zeros in \( F \).

**Proof.** Let \( a \) be a zero of \( f(x) \). Then \( 0 = f(a) = (a - c)q(a) \). Since \( F \) is an integral domain, \( a - c = 0 \) or \( q(a) = 0 \). Thus \( a = c \) or \( a \) is a zero of \( q(x) \). \( \square \)

17. Suppose that \( f(x), g(x) \) and \( h(x) \) are polynomials over the field \( F \), each of which has positive degree, and that \( f(x) = g(x)h(x) \). Prove that the zeros of \( f(x) \) in \( F \) consist of the zeros of \( g(x) \) in \( F \) together with the zeros of \( h(x) \) in \( F \).

**Proof.** Let \( a \) be a zero of \( f \). Then \( 0 = f(a) = g(a)h(a) \). Since \( F \) is an integral domain, \( g(a) = 0 \) or \( h(a) = 0 \). Thus \( a \) is a zero of \( g \) or \( a \) is a zero of \( h \).

Let \( a \) be a zero of \( g \). Then \( f(a) = g(a)h(a) = 0 \). Thus \( a \) is a zero of \( f \). Similarly if \( a \) is a zero of \( h \), then \( a \) is a zero of \( f \). \( \square \)

22. Let \( a \neq b \) in a field \( F \). Show that \( x + a \) and \( x + b \) are relatively prime in \( F[x] \).

**Proof.** Let \( a \neq b \in F \). Then \( (a - b)^{-1}(x + a) - (a - b)^{-1}(x + b) = 1 \). Therefore we have written 1 as a linear combination of \( x + a \) and \( x + b \). Thus the GCD of \( x + a \) and \( x + b \) is 1. \( \square \)

23. Let \( f(x), g(x), h(x) \in F[x] \) where \( f(x) \) and \( g(x) \) are relatively prime. If \( h(x) \mid f(x) \), prove that \( h(x) \) and \( g(x) \) are relatively prime.

**Proof.** Let \( f(x), g(x), h(x) \in F[x] \) where \( f(x) \) and \( g(x) \) are relatively prime. Assume \( h(x) \mid f(x) \). Thus there exists \( k(x) \in F[x] \) such that \( f(x) = h(x)k(x) \). Moreover, since \( f(x) \) and \( g(x) \) are relatively prime there exist \( m(x), n(x) \in F[x] \) such that \( 1 = f(x)m(x) + g(x)n(x) \). Therefore \( 1 = h(x)k(x)m(x) + g(x)n(x) \). So we have written 1 as a linear combination of \( h(x) \) and \( g(x) \). Therefore the GCD of \( h(x) \) and \( g(x) \) is 1. \( \square \)