1. Find all elements of \( \mathbb{Z}_3[x]/(p(x)) \) and construct addition and multiplication tables.

   (a) \( p(x) = x^2 + x + 2 \)
   (b) \( p(x) = x^2 + 1 \)

8. If \( F \) is a finite field with \( k \) elements, and \( p(x) \) is a polynomial of positive degree \( n \) over \( F \), find a formula for the number of elements in the ring \( F[x]/(p(x)) \).

   An arbitrary element in \( F[x]/(p(x)) \) looks like \( a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + (p(x)) \) where \( a_i \in F \).

   So we have \( k \) choices for each \( a_i \). Thus we have a total of \( k^n \) possible combinations.

   \[
   |F[x]/(p(x))| = k^n
   \]

9. Construct a field having the following number of elements.

   (a) \( 2^4 \)
   
   We want to work over \( \mathbb{Z}_2 \) and find an irreducible polynomial of degree 4.

   Let \( f(x) = x^4 + x + 1 \). If \( f(x) \) is irreducible over \( \mathbb{Z}_2 \), then \( \mathbb{Z}_2[x]/(f(x)) \) is a field with \( 2^4 \) elements.

   I’ll leave the proof that \( f(x) \) is irreducible to you.

   (b) \( 5^2 \)

   Find an irreducible polynomial over \( \mathbb{Z}_5 \) of degree 2. Then \( \mathbb{Z}_5[x]/(f(x)) \) is a field with \( 5^2 \) elements.

   (c) \( 3^3 \)

   Find an irreducible polynomial over \( \mathbb{Z}_3 \) of degree 3. Then \( \mathbb{Z}_3[x]/(f(x)) \) is a field with \( 3^3 \) elements.

   (d) \( 7^2 \)

   Find an irreducible polynomial over \( \mathbb{Z}_7 \) of degree 2. Then \( \mathbb{Z}_7[x]/(f(x)) \) is a field with \( 7^2 \) elements.