VECTOR SPACES

1 Definition

The material in this section should be a review of material learned in an introductory linear algebra class, such as Math 35. The primary difference is that in the introductory class, you most likely restricted your study to vector spaces of the real numbers or, possibly, over the complex numbers. That is, you used \( \mathbb{R} \) or \( \mathbb{C} \) as your field of scalars. In this class, we will allow any field to serve as our field of scalars. So, for example, we can study vector spaces over \( \mathbb{Z}_5 \).

**Definition 1** A set \( V \) is said to be a **vector space over** a field \( F \) if \( V \) is an Abelian group under addition and if for each \( a \in F \) and \( v \in V \), there is an element \( av \in V \) such that the following conditions hold: for all \( a, b \in F \) and all \( u, v \in V \):

1. \( a (v + u) = av + au \)
2. \( (a + b) v = av + bv \)
3. \( a (bv) = (ab) v \)
4. \( 1_F v = v \)

Recall that the element of \( V \) (i.e. the members of the vector space) are called **vectors**, while the elements of \( F \) are called **scalars**. Also, the product \( av \) of a scalar, \( a \), and a vector, \( v \), is called scalar multiplication. There is NO vector multiplication in the definition of a vector space.

**Example 2** For any natural number \( n \) and any field \( F \), \( M_n (F) \), the set of \( n \times n \) matrices with entries from \( F \) is a vector space over \( F \). (under the standard operations of addition of matrices and scalar multiplication)

**Example 3** For any field, \( F \), the set of polynomials, \( F [x] \), is a vector space over \( F \).

**Example 4** For any field, \( F \), the set \( F^n = \{ (a_1, a_2, \ldots, a_n) : a_i \in F \} \) is a vector space over \( F \) under the following operations:

\[
(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)
\]

\[
c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)
\]

**Example 5** For any field, \( E \), if \( F \) is a subfield of \( E \), then \( E \) is a vector space over \( F \). (under the operations of \( E \))

**Definition 6** Let \( V \) be a vector space over a field \( F \) and let \( U \) be a subset of \( V \). We say that \( U \) is a **subspace** of \( V \) if \( U \) is also a vector space over \( F \) under the operations of \( V \).

**Lemma 7** Let \( V \) be a vector space over a field \( F \) and let \( U \) be a subset of \( V \). Then \( U \) is a subspace of \( V \) if and only if for all \( u, v \in U \) and \( a \in F \)
1. $U \neq \emptyset$
2. $u + v \in U$, and
3. $au \in U$.

A homomorphism (i.e. operation preserving mapping) between two vector spaces is referred to as a linear transformation.

**Definition 8** Let $V$ and $W$ be vector spaces over a field $F$. A mapping $T : V \to W$ is said to be a **linear transformation** if $T$ preserves vector addition and scalar multiplication. That is,

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$, and
2. $T(au) = aT(u)$ for all $a \in F$ and all $u \in V$.

A one-to-one linear transformation from $V$ onto $W$ is called a **vector space isomorphism**.

## 2 Linear Independence and Spanning Sets

**Definition 9** A set $S$ of vectors is said to be **linearly dependent** over the field $F$ if there are vectors $v_1, v_2, \ldots, v_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zero, such that $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. A set of vectors that is not linearly dependent over $F$ is called **linearly independent** over $F$.

**Definition 10** A set $S$ of vectors is said to **span** the vector space $V$ if every element of $V$ is a linear combination of vectors in $S$ with coefficients from $F$.

**Definition 11** Let $V$ be a vector space over $F$. A subset $B$ of $V$ is called a **basis** for $V$ if

1. $B$ is linearly independent over $F$ and
2. $B$ spans $V$.

**Example 12** $\mathbb{Q}[x]$ is a vector space over $\mathbb{Q}$ with basis $\{1, x, x^2, x^3, \ldots\}$.

## 3 Dimension

**Definition 13** Let $V$ be a vector space. Then, the **dimension** of $V$, denoted $\dim V$, is equal to the size of a basis for $V$.

**Remark 14** **IMPORTANT!!!** For the definition above to make sense, we need to know that every basis has the same number of vectors. The following two theorems verify this fact.

**Theorem 15** If a vector space has one basis that contains infinitely many elements, then every basis contains infinitely many vectors.
**Proof.** Homework. ■

**Theorem 16** Let $V$ be a vector space. Suppose $V$ has a finite basis $\{v_1, v_2, \ldots, v_n\}$. Then every basis of $V$ contains $n$ vectors.

**Proof.** By theorem 15, $V$ does not have an infinite basis.
Let $\{w_1, w_2, \ldots, w_m\}$ be another basis for $V$. WLOG assume $n \leq m$.
Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for $V$ and $w_1 \in V$, we have $w_1 = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for some $a_i \in F$.

Note that there must be at least one scalar, $a_i$, not equal to zero, since $w_1 \neq 0$. WLOG assume $a_1 \neq 0$.

Then $a_1^{-1} w_1 = v_1 + a_1^{-1} a_2 v_2 + \cdots + a_1^{-1} a_n v_n$. So, $v_1 = c_1 w_1 + c_2 v_2 + \cdots + c_n v_n$ where $c_1 = a_1^{-1}$ and $c_i = -a_1^{-1} a_i$ for $i \geq 2$.

**Claim:** $\{w_1, v_2, \ldots, v_n\}$ spans $V$.

Let $u \in V$. Then $u = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$ for some $b_i \in F$, since $\{v_1, v_2, \ldots, v_n\}$ is a basis for $V$. So,

$$u = b_1 (c_1 w_1 + c_2 v_2 + \cdots + c_n v_n) + b_2 v_2 + \cdots + b_n v_n$$

$$= d_1 w_1 + d_2 v_2 + \cdots + d_n v_n$$

where $d_1 = b_1 c_1$ and $d_i = b_1 c_i + b_i$ for $i \geq 2$.
Thus, $u$ is in the span of $\{w_1, v_2, \ldots, v_n\}$. So, $\{w_1, v_2, \ldots, v_n\}$ spans $V$.

So, since $w_2 \in V$, we have $w_2 = k_1 w_1 + k_2 v_2 + \cdots + k_n v_n$ for some $k_i \in F$.

Note that, at least one of $k_2, k_3, \ldots, k_n$ must be nonzero. If not then we would have $w_2 = k_1 w_1$ and this would contradict the fact that $\{w_1, w_2, \ldots, w_m\}$ is a basis (and thus a linearly independent set).

WLOG assume $k_2 \neq 0$.

From this we can prove, similarly to that above, that $\{w_1, w_2, v_3, \ldots, v_n\}$ spans $V$.
Continuing in this way we get that $\{w_1, w_2, \ldots, w_n\}$ spans $V$.

If $n \neq m$, then $w_{n+1}$ is a linear combination of $w_1, w_2, \ldots, w_n$, so $\{w_1, \ldots, w_n\}$ is not linearly independent. This is a contradiction since $\{w_1, w_2, \ldots, w_m\}$ is a basis.

Therefore, $n = m$. ■

4 **Homework**

1. Prove lemma 7

2. In $\mathbb{R}[x]$ consider the set $V = \{a_2 x^2 + a_1 x + a_0 : a_0, a_1, a_2 \in \mathbb{R}\}$.
   (a) Prove that $V$ is a subspace of $\mathbb{R}[x]$.
   (b) Find a basis for $V$.
   (c) Is $\{x^2 + x + 1, x + 5, 3\}$ a basis?

3. Let $V = \mathbb{R}^3$ and let $W = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 = c^2\}$. Is $W$ a subspace of $V$? If so, what is its dimension?
4. Let \( V = \mathbb{R}^3 \) and let \( W = \{(a, b, c) \in \mathbb{R}^3 : a + b = c \} \). Is \( W \) a subspace of \( V \)? If so, what is its dimension?

5. Prove that every spanning set contains a basis: If \( \{v_1, v_2, \ldots, v_n\} \) is a spanning set for a vector space \( V \), then some subset of \( V \) is a basis for \( V \).

6. Prove that every linearly independent set can be extended to a basis: If \( \{v_1, v_2, \ldots, v_n\} \) is a linearly independent set in a vector space \( V \), then there exist vectors, \( w_1, w_2, \ldots, w_m \) in \( V \) such that \( \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m\} \) is a basis for \( V \).

7. If \( V \) is a vector space over \( F \) of dimension 5 and \( U \) and \( W \) are subspaces of \( V \) of dimension 3, prove that \( U \cap W \neq \{0\} \).

8. IMPORTANT!!! Let \( \{v_1, v_2, \ldots, v_n\} \) be a finite set of vectors in a vector space \( V \). Prove that \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( V \) if and only if every member of \( V \) can be written uniquely as a linear combination of the vectors in \( \{v_1, v_2, \ldots, v_n\} \).

9. Let \( V \) be a vector space over \( \mathbb{Z}_p \) with \( \dim V = n \). How many elements are in \( V \)? (hint: use #8)

10. Let \( V \) be a finite dimensional vector space. Let \( U \) be a subspace of \( V \).

    (a) Prove \( \dim U \leq \dim V \).

    (b) Prove that \( \dim U < \dim V \), if \( U \neq V \).


12. Let \( T \) be a linear transformation of \( V \) onto \( W \) (where \( V \) and \( W \) are both vector spaces over a field \( F \)). If \( \{v_1, v_2, \ldots, v_n\} \) spans \( V \), show that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) spans \( W \).

13. Let \( V \) be a vector space over a field \( F \) with \( \dim V = n \). Prove that \( V \) is isomorphic to \( F^n \).