28. (a) Let $A$ be an $m \times n$ matrix with a row consisting entirely of zeros. Show that if $B$ is an $n \times p$ matrix, then $AB$ has a row of zeros.

**Proof.** Let $A = [a_{ij}]$ and assume that the $l$th row of $A$ is all zeros. So, $a_{lj} = 0$ for all $j$ between 1 and $n$. Let $B = [b_{ij}]$. Then $AB = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. We claim that the $l$th row of $AB$ is zero, so we need to show that $c_{lj} = 0$ for all $j$ between 1 and $n$.

$$c_{lj} = \sum_{k=1}^{n} a_{lk}b_{kj} = \sum_{k=1}^{n} (0)b_{kj} = 0$$

Thus, the $l$th row of $AB$ is zeros. \[\square\]

(b) Let $A$ be an $m \times n$ matrix with a column consisting entirely of zeros and let $B$ be $p \times m$. Show that $BA$ has a column of zeros.

**Proof.** Let $A = [a_{ij}]$ and assume that the $l$th column of $A$ is all zeros. So, $a_{il} = 0$ for all $i$ between 1 and $n$. Let $B = [b_{ij}]$. Then $BA = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$. We claim that the $l$th column of $BA$ is zero, so we need to show that $c_{il} = 0$ for all $i$ between 1 and $n$.

$$c_{il} = \sum_{k=1}^{n} b_{ik}a_{kl} = \sum_{k=1}^{n} b_{ik}(0) = 0$$

Thus, the $l$th column of $BA$ is zeros. \[\square\]

51. Let $x$ be an $n$-vector.

(a) Is it possible for $x \cdot x$ to be negative? Explain.

Let $x$ be an $n$-vector, so $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then $x \cdot x = x_1^2 + x_2^2 + \cdots + x_n^2$. And since $x_1, x_2, \ldots, x_n$ are all real numbers, then the squares of them are positive. So we are adding a bunch of non-negative numbers together and hence we must get a non-negative number.

(b) If $x \cdot x = 0$, what is $x$?

Since the dot product shown above is $x_1^2 + x_2^2 + \cdots + x_n^2$, where each term is non-negative, there will never be any cancellation. Thus the only way for this sum to be zero is if each term is zero. Hence $x$ must equal the $0$ vector.