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WEAK BOURBAKI UNMIXED RINGS: A STEP TOWARDS NON-NOETHERIAN COHEN-MACAULAYNESS

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ABSTRACT. Weak Bourbaki unmixed rings are defined in this paper. The definition of a weak Bourbaki unmixed ring is a candidate for an "appropriate" definition of Cohen-Macaulayness. We will see that this definition satisfies many of the conditions we want an "appropriate" definition to satisfy. It is not yet known whether this definition (or any other) satisfies all of the conditions. However, no example has been found of a weak Bourbaki unmixed ring which violates one of the conditions.

1. Introduction. Cohen-Macaulay rings have been studied since 1916 when Macaulay's *The algebraic theory of modular systems* [**10**] was published. At the time, Macaulay was dealing with the question of describing solutions to sets of polynomial equations. Since that time the theory of Cohen-Macaulay rings (and modules) has grown to be one of the major areas of study in commutative algebra and in algebraic geometry, see [**1**].

It was not until 1992 that anyone brought up the idea of extending the concept of Cohen-Macaulayness to non-Noetherian rings. In 1992, Glaz published a paper in which she studied fixed rings [**3**]. At the end of that paper there was a conjecture that under certain conditions these fixed rings would be Cohen-Macaulay. However, since the rings Glaz was studying were not Noetherian, this conjecture cannot be considered until there is an appropriate definition of non-Noetherian Cohen-Macaulayness. In fact, Glaz goes on to say that the "first step toward solving this conjecture is finding the right definition of non-Noetherian Cohen-Macaulayness" [**3**].

In 1994 [**4**], Glaz refined this question by asking whether one can define a non-Noetherian Cohen-Macaulay ring so that:

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G1. The definition coincides with the usual notion if the ring is Noetherian, and

G2. regular rings, at least when coherent, are Cohen-Macaulay.

Glaz considered using the equality of the depth and the dimension of a ring as the definition of non-Noetherian Cohen-Macaulayness. She was able to give an example, however, of a class of coherent regular rings which are not Cohen-Macaulay in this sense [**4**].

In preparing to answer Glaz's question, I have considered what other conditions it would be desirable for a definition of non-Noetherian Cohen-Macaulayness to satisfy. In particular, I propose that the following two conditions should be satisfied by non-Noetherian Cohen-Macaulayness:

H1. R is Cohen-Macaulay if and only if $R[X]$ is Cohen-Macaulay, and

H2. R is Cohen-Macaulay if and only if $R_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p}$ in R.

The condition H2 is especially important since so much of what is done in commutative algebra is done by reducing a general question about commutative rings to one about local rings.

In Section 2 the definition of a weak Bourbaki unmixed ring is given as is the usual definition of (Noetherian) Cohen-Macaulayness. The definition of weak Bourbaki unmixed is one candidate for the definition of non-Noetherian Cohen-Macaulayness. In Section 3 it will be shown that this definition satisfies condition G1 as well as half of each of conditions H1 and H2. Section 4 shows that the ring $k[X_1, X_2, X_3, \cdots]$, where k is a field, is weak Bourbaki unmixed. This is important since this ring is a coherent regular ring and condition G2 states that our definition of non-Noetherian Cohen-Macaulayness should satisfy the condition that all coherent regular rings are Cohen-Macaulay. Finally, Section 5 includes some results on low-dimensional rings and some examples.

It is not yet known whether weak Bourbaki unmixedness satisfies G2 and the other halves of H1 and H2. However, no counterexamples have been found, so this is still an open question.

2. Definitions and preliminary results. Throughout this paper, R denotes a commutative ring with unity. We will use the notation $\mathrm{Min}\,(I)$ to denote the set of all minimal prime overideals, also called minimal prime divisors, of the ideal I. That is, $\mathrm{Min}\,(I)$ is the set of all prime ideals in the ring which contain the ideal I but do not contain any other prime ideal which contains I. Also, the notation $\mathrm{Ass}_f(I)$ will denote the set of weak Bourbaki associated prime divisors of the ideal I. That is, $\mathrm{Ass}_f(I)$ is the set of all prime ideals which are in $\mathrm{Min}\,(I:a)$ for some element a of the ring.

In Macaulay's original work which led to the theory of Cohen-Macaulay rings [**10**], Macaulay was studying the unmixedness of certain ideals. He was studying those ideals I which can be generated by $\mathrm{ht}\, I$ elements. If $\mu(I)$ is used to denote the minimal number of generators needed for the ideal I, this can be restated by saying that he was studying those ideals I which satisfied the condition $\mu(I) \leq \mathrm{ht}\, I$. These types of ideals play a central role in what is to follow.

Definition 1. A finitely generated ideal I will be said to be *height-generated* if $\mu(I) \leq \mathrm{ht}\, I$.

Note that, if R is Noetherian, then $\mu(I) \leq \mathrm{ht}\, I$ is equivalent to $\mu(I) = \mathrm{ht}\, I$ by Krull's principal ideal theorem.

Now we are ready to define weak Bourbaki unmixed rings. From this point on, weak Bourbaki unmixed will be abbreviated by wB-unmixed.

Definition 2. A commutative ring with unity will be said to be wB-*unmixed* if each height-generated ideal in the ring has the property that none of its weak Bourbaki associated prime ideals are embedded. That is, if each height-generated ideal I in the ring has the property

$$\mathrm{Ass}_f(I) = \mathrm{Min}\,(I).$$

The following lemma follows easily from the definition of $\mathrm{Ass}_f(I)$ and from the definition of wB-unmixed.

Lemma 1. *Let R be a commutative ring with unity. Then R is* wB-*unmixed if and only if every height-generated ideal I in R satisfies*

$$\mathrm{Min}\,(I:a) \subseteq \mathrm{Min}\,(I) \quad \textit{for all} \quad a \notin I.$$

This lemma provides a very useful characterization of wB-unmixed rings. This characterization will be used often in the proofs that follow.

3. wB-unmixed rings. Now that wB-unmixed rings have been defined, we want to determine which, if any, of our four conditions, i.e., G1, G2, H1 and H2, for non-Noetherian Cohen-Macaulayness are satisfied by wB-unmixedness. The three theorems that follow in this section will give us that wB-unmixedness satisfies G1 as well as half of each of H1 and H2. In Section 4 there will be a result relating wB-unmixedness and condition G2.

At this point it may be helpful to recall the definition of a Cohen-Macaulay ring in the usual (Noetherian) setting. A Cohen-Macaulay ring is a Noetherian ring in which every height-generated ideal is height-unmixed. That is, each height-generated ideal in the ring has the property that all of its associated prime ideals have the same height as the original ideal. Note that it is not necessary to specify which set of associated primes is being used here (Noether, weak Bourbaki, Zariski-Samuel, etc.) since the ring is Noetherian.

Recall that the first condition for non-Noetherian Cohen-Macaulayness, G1, is that the condition should be equivalent to the usual definition of Cohen-Macaulayness when the ring is Noetherian. The following theorem shows that wB-unmixed rings satisfy this condition.

Theorem 1. *If R is a Noetherian commutative ring, then R is a* wB*-unmixed ring if and only if R is a Cohen-Macaulay ring.*

Proof. Clearly it is sufficient to prove that a height-generated ideal in a Noetherian ring is height-unmixed if and only if that ideal has no weak Bourbaki embedded components.

This follows easily from the fact that, in a Noetherian ring, every minimal prime overideal of an ideal which can be generated by r elements has a height of no more than r, see [**11**], and the fact that every prime overideal of an ideal has a height which is greater than or equal to the height of the original ideal, by definition of height. $\square$

Now recall condition H1:

H1. R is Cohen-Macaulay if and only if $R[X]$ is Cohen-Macaulay.

The following theorem shows that wB-unmixed rings satisfy at least half of this condition.

Theorem 2. *Let R be a commutative ring with unity. If $R[X]$ is a* wB-*unmixed ring, then R is a* wB-*unmixed ring.*

The following two lemmas will be used to prove Theorem 2. The lemmas will be proved in Section 6.

Lemma 2. *Let R be a commutative ring with unity, and let I be an ideal of finite height in R. Let J be the ideal in $R[X]$ generated by the elements of I and the element X, so $J = (I, X)$. Then*

$$\operatorname{ht} J \geq 1 + \operatorname{ht} I.$$

From this lemma the result follows that if I is a height-generated ideal in R, then $J = (I, X)$ is a height-generated ideal in $R[X]$.

Lemma 3. *Let R be a commutative ring with unity, and let I be an ideal in R. Let $J = (I, X)$ as in the previous lemma. If* $\operatorname{Min}(J : g) \subseteq \mathit{Min}\, J$ *for all $g \notin J$, then* $\operatorname{Min}(I : a) \subseteq \mathit{Min}\, I$ *for all $a \notin I$.*

Now Theorem 2 can be proven.

Proof. Let R be a commutative ring with unity such that $R[X]$ is a wB-unmixed ring. Let I be a height-generated ideal in the ring R. Let $J = (I, X)$ be the corresponding ideal in $R[X]$. Then, by, Lemma 2, J is height-generated. So, since $R[X]$ is assumed to be wB-unmixed, it follows that $\operatorname{Min}(J : g) \subseteq \operatorname{Min} J$ for all $g \notin J$. So by Lemma 3, $\operatorname{Min}(I : a) \subseteq \operatorname{Min} I$ for all $a \notin I$. Therefore, R is wB-unmixed. $\square$

Finally, recall condition H2:

H2. R is Cohen-Macaulay if and only if $R_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p}$ in R.

The following theorem shows that wB-unmixed rings also satisfy at least half of this condition.

Theorem 3. *Let R be a commutative ring with unity. If $R_{\mathfrak{p}}$ is a* wB-*unmixed ring for all prime ideals $\mathfrak{p}$ in R, then R is a* wB-*unmixed ring.*

The following two lemmas will be needed to prove this theorem. These lemmas will also be proven in Section 6.

Lemma 4. *Let I be an ideal in a commutative ring R with unity. Let $\mathfrak{m}$ be a maximal ideal in R, and let $J = IR_{\mathfrak{m}}$ be the corresponding ideal in the localization $R_{\mathfrak{m}}$. Then, for any $a \in R$,*

$$(J : a) = (I : a)R_{\mathfrak{m}}.$$

Lemma 5. *Let R be a commutative ring with unity, and let I be an ideal in R. Let $\mathfrak{q} \in \operatorname{Min}(I : a)$ for some $a \notin I$, and let $\mathfrak{m}$ be a maximal ideal in R such that $\mathfrak{q} \subseteq \mathfrak{m}$. Let $J = IR_{\mathfrak{m}}$. If $\operatorname{Min}(J : a) \subseteq \operatorname{Min} J$, then $\mathfrak{q}$ is minimal over I.*

Now the proof of Theorem 3.

Proof. Let R be a commutative ring with unity such that $R_{\mathfrak{p}}$ is wB-unmixed for all prime ideals $\mathfrak{p}$ in R. Let I be a height-generated ideal in the ring R, and let a be an element of R such that $a \notin I$. To show that R is wB-unmixed, it must be shown that $\operatorname{Min}(I : a) \subseteq \operatorname{Min} I$. For this, let $\mathfrak{q}$ be an element of $\operatorname{Min}(I : a)$, and let $\mathfrak{m}$ be a maximal ideal in R which contains $\mathfrak{q}$.

Let $J = IR_{\mathfrak{m}}$. Then, using the correspondence between primes in R which are contained in $\mathfrak{m}$ and primes in $R_{\mathfrak{m}}$ and the fact that $I \subseteq \mathfrak{m}$, it follows that $\operatorname{ht} J \geq \operatorname{ht} I$. Note also that J is generated in $R_{\mathfrak{m}}$ by the images of the elements that generated I in R under the natural homomorphism from R to $R_{\mathfrak{m}}$. Therefore, J is height-generated. So, since $R_{\mathfrak{m}}$ is assumed to be wB-unmixed, $\operatorname{Min}(J : g) \subseteq \operatorname{Min} J$ for all $g \notin J$.

Clearly, $a \notin J$ since otherwise $(J : a) = R_{\mathfrak{m}}$, but $(J : a) = (I : a)R_{\mathfrak{m}}$ by Lemma 4, and $(I : a) \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ implies $(I : a)R_{\mathfrak{m}} \subseteq \mathfrak{m}R_{\mathfrak{m}} \neq R_{\mathfrak{m}}$, so $(J : a) \neq R_{\mathfrak{m}}$. Thus, $\operatorname{Min}(J : a) \subseteq \operatorname{Min} J$. So, by Lemma 5, $\mathfrak{q}$ is minimal over I. Therefore, $\operatorname{Min}(I : a) \subseteq \operatorname{Min} I$. $\square$

Note that, in this theorem, it is not necessary to assume that $R_{\mathfrak{p}}$ is wB-unmixed for every prime ideal $\mathfrak{p}$ in R. It is sufficient to assume that $R_{\mathfrak{m}}$ is wB-unmixed for every maximal ideal $\mathfrak{m}$ in R.

4. $k[X_1, X_2, X_3, \dots]$**.** The goal of this section is to show that $k[X_1, X_2, X_3, \dots]$ is wB-unmixed for every field k. This result is of interest for two reasons. First it is of historical importance since the polynomial rings $k[X_1, X_2, \dots , X_n]$ where k is either the field of real numbers or is the field of complex numbers where the theory of Cohen-Macaulay rings began with Macaulay's study of unmixedness properties [**10**]. Second, it is of importance because the ring $k[X_1, X_2, X_3, \dots]$ where k is a field is a coherent regular ring, so this result is related to condition G2 which is a requirement for non-Noetherian Cohen-Macaulayness.

Theorem 4. *Let R be a Cohen-Macaulay domain. Then $R[X_1, X_2, X_3, \dots]$ is* wB*-unmixed.*

From this theorem the desired result follows since every field is Cohen-Macaulay.

Corollary 1. *For any field k, the ring $k[X_1, X_2, X_3, \dots]$ is a* wB*-unmixed ring.*

In order to prove Theorem 4, the following lemmas regarding the relationships between heights of prime ideals in R and prime ideals in $R[X_1, X_2, \dots]$ will be used. The proofs of these lemmas can be found in Section 6.

Lemma 6. *Let R be a Noetherian domain, and let $\mathfrak{p}$ be a prime ideal in R. Let $R' = R[X_1, X_2, X_3, \dots]$, and for each integer $i \geq 1$, let*

$R_i = R[X_1, X_2, \dots, X_i]$. *Then*

(1) *For each* $i \geq 1$, *the ideal* $\mathfrak{p}_i = \mathfrak{p}R_i$ *in* R_i *has the same height in* R_i *as does* $\mathfrak{p}$ *in* R. *That is,*

$$\operatorname{ht} \mathfrak{p}_i = \operatorname{ht} \mathfrak{p}.$$

(2) *The ideal* $\mathfrak{p}' = \mathfrak{p}R'$ *in* R' *has the same height in* R' *as does* $\mathfrak{p}$ *in* R. *That is,*

$$\operatorname{ht} \mathfrak{p}' = \operatorname{ht} \mathfrak{p}.$$

Lemma 7. *Let* R *be a commutative ring. Let* S *be a polynomial ring over* R *of the form* $R[X_1, X_2, \dots, X_i]$ *for some* $i \geq 1$, *or* $R[X_1, X_2, X_3, \dots]$. *Let* J *be an ideal in* R, *and let* I *be the ideal in* S *generated by* J, *that is,* $I = JS$. *Finally, let* a *be an element of* R. *Then*

$$(I : a) = (J : a)S.$$

Now the proof of Theorem 4.

Proof. Let R be Cohen-Macaulay and, for every integer $i \geq 1$, let $R_i = R[X_1, X_2, \dots, X_i]$. Since R is Cohen-Macaulay, each R_i is Cohen-Macaulay. Let $S = R[X_1, X_2, X_3, \dots]$. Then $S = \varinjlim R_i$. Let I be a height-generated ideal in S with $\operatorname{ht} I = N$. Let a be an element of S which is not in I. It must be shown that $\operatorname{Min}(I : a) \subseteq \operatorname{Min} I$. For this, take $\mathfrak{p} \in \operatorname{Min}(I : a)$ and assume $\mathfrak{q}$ is a prime ideal in S with $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$.

Since I is height-generated with height equal to N, there exist N elements, $a_1, a_2, \dots, a_N$ in S such that $I = (a_1, a_2, \dots, a_N)$. Now since $S = \varinjlim R_i$, there is some positive integer M such that $\{a_1, a_2, \dots, a_N, a\} \subseteq R_i$ for all $i \geq M$. For each $i \geq M$, let J_i be the ideal generated by $\{a_1, a_2, \dots, a_N\}$ in R_i. Then $I = J_i S$.

Claim. $\operatorname{ht} J_i = N$ *for all* $i \geq M$.

First note that, since R_i is Noetherian and since J_i can be generated by N elements, $\operatorname{ht} J_i \leq N$ by Krull's principal ideal theorem.

On the other hand, for any prime ideal $\mathfrak{n}$ in R_i with $J_i \subseteq \mathfrak{n}$ and $\operatorname{ht}\mathfrak{n} = \operatorname{ht} J_i$, the ideal $\mathfrak{s} = \mathfrak{n}S$ in S can be constructed and $\mathfrak{s}$ will be a prime ideal in S. Also, by Lemma 6, $\operatorname{ht}\mathfrak{s} = \operatorname{ht}\mathfrak{n}$. Finally, since J_i is contained in $\mathfrak{n}$, it follows that $J_iS \subseteq \mathfrak{n}S$, that is, $I \subseteq \mathfrak{s}$. So, $\operatorname{ht} I \leq \operatorname{ht}\mathfrak{s}$. Putting this together with the fact that the heights of $\mathfrak{s}, \mathfrak{n}$ and J_i are all equal, it follows that $\operatorname{ht} I \leq \operatorname{ht} J_i$. That is, $N \leq \operatorname{ht} J_i$.

Putting these together gives the desired result of $\operatorname{ht} J_i = N$, so the claim has been proven.

J_i can be generated by $N = \operatorname{ht} J_i$ elements, namely $a_1, a_2, \dots, a_N$, so J_i is height-generated. Thus, since R_i is Cohen-Macaulay (and is thus wB-unmixed) $\operatorname{Min}(J_i : g) \subseteq \operatorname{Min} J_i$ for all $g \notin J_i$, where $g \in R_i$. In particular, since a is an element of R_M which is not in J_i, it follows that $\operatorname{Min}(J_i : a) \subseteq \operatorname{Min} J_i$.

Recall that at the beginning of this proof prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ were chosen in S with $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \operatorname{Min}(I : a)$. Now define ideals $\mathfrak{p}_i = \mathfrak{p} \cap R_i$ and $\mathfrak{q}_i = \mathfrak{q} \cap R_i$ in R_i. Both $\mathfrak{p}_i$ and $\mathfrak{q}_i$ are prime ideals in R_i. Also, since $J_i \subseteq I \cap R_i$, it follows that $J_i \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$.

Claim. $\mathfrak{p}_i \in \operatorname{Min}(J_i : a)$.

Once this claim is proven, it will follow that $\mathfrak{p}_i \in \operatorname{Min} J_i$, since $\operatorname{Min}(J_i : a) \subseteq \operatorname{Min} J_i$, from which it will follow that $\mathfrak{p}_i = \mathfrak{q}_i$. Since this will be true for all $i \geq M$ and since $\mathfrak{p}$ and $\mathfrak{q}$ are the direct limits of the $\mathfrak{p}_i$ and $\mathfrak{q}_i$, respectively, it will follow that $\mathfrak{p} = \mathfrak{q}$ which will finish the proof of the theorem.

In order to prove this claim, it must first be shown that $(J_i : a) \subseteq \mathfrak{p}_i$. For this, let x be in $(J_i : a)$. Then $xa \in J_i$. Since $J_i \subseteq I$, this gives $xa \in I$, so $x \in (I : a)$. Finally, $(I : a) \subseteq \mathfrak{p}$, so $x \in \mathfrak{p}$. So, since x is also an element of R_i, it follows that $x \in \mathfrak{p} \cap R_i = \mathfrak{p}_i$.

Now let $\mathfrak{a}$ be a prime ideal in R_i with $(J_i : a) \subseteq \mathfrak{a} \subseteq \mathfrak{p}_i$. Let $\mathfrak{a}' = \mathfrak{a}S$. Then $\mathfrak{a}'$ is a prime ideal in S. By Lemma 7, $(I : a) = (J_i : a)S$. So,

$$(I : a) = (J_i : a)S \subseteq \mathfrak{a}S = \mathfrak{a}'.$$

Also, $\mathfrak{a}' \subseteq \mathfrak{p}$, so

$$(I : a) \subseteq \mathfrak{a}' \subseteq \mathfrak{p}.$$

However, $\mathfrak{p}$ was chosen to be minimal over $(I : a)$, so it must be that $\mathfrak{a}' = \mathfrak{p}$. So $\mathfrak{a}' \cap R_i = \mathfrak{p} \cap R_i$. Thus $\mathfrak{a} = \mathfrak{p}_i$. Therefore, $\mathfrak{p}_i$ is minimal over $(J_i : a)$. $\square$

5. Low-dimensional rings and examples. First, an example of a ring which is not wB-unmixed.

Example 1. Let $R = k[X, XY, XY^2, XY^3, \dots]$ where k is a field. Then R is not a wB-unmixed ring.

To see this, let $I = (XY)$. Then $\operatorname{ht} I = 1$, so I is height-generated. Let $a = XY^2$. Then a is not an element of I.

$$(I : a) = (X, XY, XY^2, XY^3, \dots)$$

is a prime ideal in R, so it is an associated prime to I. However,

$$I \subset (XY, XY^2, XY^3, \dots) \subset (X, XY, XY^2, XY^3, \dots)$$

and $(XY, XY^2, XY^3, \dots)$ is a prime ideal in R, so $(X, XY, XY^2, XY^3, \dots)$ is an associated prime to I which is not minimal over I. Thus, R is not a wB-unmixed ring.

It is true that any zero-dimensional Noetherian ring is Cohen-Macaulay. The following lemma gives the same result for non-Noetherian rings with regard to wB-unmixedness.

Lemma 8. *Any zero-dimensional commutative ring is* wB-*unmixed.*

This lemma follows, as does the result for Noetherian rings and Cohen-Macaulayness, simply from the fact that, in a zero-dimensional ring, there can be no embedded components for any ideal.

Example 2. Let k be a field, and let

$$S = k[X, Y_1, Y_2, \dots]/(X^2, XY_i, Y_iY_j)_{i,j \geq 1}.$$

Then S is wB-unmixed, since $\dim S = 0$.

For one-dimensional rings to be assured of being wB-unmixed, stronger hypotheses are needed.

Theorem 5. *Any one-dimensional commutative domain is* wB-*unmixed.*

Proof. In a one-dimensional domain, every nonzero ideal has height equal to one. Since these ideals have height equal to the dimension of the ring, they cannot have any embedded components. Therefore, to prove this theorem, it is only necessary to show that the zero ideal has no embedded components. That is, it must be shown that in a one-dimensional commutative domain, $\mathrm{Min}\,((0) : a) \subseteq \mathrm{Min}\,(0)$ for every nonzero element a of the ring. To see that this is true, note that if a is a nonzero element of the domain R, then

$$\begin{aligned}((0) : a) &= \{r \in R \mid ra \in (0)\}\\ &= \{r \in R \mid ra = 0\}\\ &= (0).\end{aligned}$$

Therefore, $\mathrm{Min}\,((0) : a) = \mathrm{Min}\,(0)$ for every $a \notin (0)$. $\square$

Example 3. Let k be a field, and let

$$R = k[X, Y_1, Y_2, \ldots]/(XY_i, Y_iY_j)_{i,j\geq 1}.$$

Then R is not wB-unmixed.

Before we see why R is not wB-unmixed, note that R is one-dimensional. It is not, however, a domain. Thus, this example gives us that the condition that the ring be a domain in the previous theorem was a necessary condition.

Now, to see why R is not wB-unmixed, consider the zero ideal in R.

$$((0) : Y_1) = (X, Y_1, Y_2, Y_3, \ldots)$$

which is a prime ideal in R, so $(X, Y_1, Y_2, Y_3, \ldots)$ is an associated prime to (0). It is not, however, minimal over (0) since $(Y_1, Y_2, Y_3, \ldots)$ is also a prime ideal in R and

$$(0) \subset (Y_1, Y_2, Y_3, \ldots) \subset (X, Y_1, Y_2, Y_3, \ldots).$$

6. Proofs of lemmas. This section only contains the proofs of the lemmas used in Sections 3 and 4.

Proof of Lemma 2. This lemma follows immediately from Theorem 38 of Kaplansky's *Commutative rings* [**7**, p. 26].

Proof of Lemma 3. Let I be an ideal in R, and let J be the ideal (I, X) in $R[X]$. Assume that $\mathrm{Min}\,(J : g) \subseteq \mathrm{Min}\,J$ for all $g \notin J$. Let a be an element of R such that $a \notin I$. It must be shown that $\mathrm{Min}\,(I : a) \subseteq \mathrm{Min}\,I$. For this, let $\mathfrak{q}$ be minimal over $(I : a)$. Since $I \subseteq (I : a)$ and $(I : a) \subseteq \mathfrak{q}$, it follows that $I \subseteq \mathfrak{q}$. Let $\mathfrak{p}$ be a prime ideal in R such that $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$. Now let $\mathfrak{p}' = (\mathfrak{p}, X)$ and $\mathfrak{q}' = (\mathfrak{q}, X)$ the corresponding prime ideals in $R[X]$. Clearly, since $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$,

$$J \subseteq \mathfrak{p}' \subseteq \mathfrak{q}'.$$

To finish this proof an element $g \notin J$ will be exhibited with the property that $\mathfrak{q}'$ is minimal over $(J : q)$ which will mean that $\mathfrak{q}' \in \mathrm{Min}\,J$ since it has been assumed that $\mathrm{Min}\,(J : g) \subseteq \mathrm{Min}\,J$. This will imply that $\mathfrak{p}' = \mathfrak{q}'$ so $\mathfrak{p} = \mathfrak{q}$. Finally this will show that $\mathfrak{q}$ is minimal over I and the desired result will be proven.

Recall that at the beginning of this proof an element $a \notin I$ for which $\mathfrak{q} \in \mathrm{Min}\,(I : a)$ was chosen. Now, let $g \in R[X]$ be the constant polynomial $g \equiv a$. Then $g \notin J$ since J consists of those polynomials whose constant term is in I.

Claim. $\mathfrak{q}'$ *is minimal over* $(J : g)$.

First it must be verified that $(J : g)$ is in fact contained in $\mathfrak{q}'$. To see this, let f be an element of $(J : g)$; then $fg \in J$, so the constant term $(fg)(0)$ of fg is in I. Since $(fg)(0) = f(0)g(0) = f(0) \cdot a$, this implies that $f(0) \cdot a \in I$, so $f(0) \in (I : a)$. Finally, using the fact that $(I : a) \subseteq \mathfrak{q}$, it follows that the constant term, $f(0)$, of f is in $\mathfrak{q}$. Therefore, $f \in \mathfrak{q}'$.

Now to see that $\mathfrak{q}'$ is in fact minimal over $(J : g)$ assume that $\mathfrak{s}$ is a prime ideal in $R[X]$ with

$$(J : g) \subseteq \mathfrak{s} \subseteq \mathfrak{q}'.$$

Then, by the correspondence of primes in R and primes in $R[X]$ which contain (X) and, since $X \in J \subseteq (J : g)$,

$$(I : a) \subseteq \mathfrak{n} \subseteq \mathfrak{q}$$

in R where $\mathfrak{n}$ is the prime ideal in R corresponding to $\mathfrak{s}$. Since $\mathfrak{q} \in \operatorname{Min}(I : a)$ and $\mathfrak{n}$ is prime, it must be that $\mathfrak{n} = \mathfrak{q}$. So

$$\mathfrak{q}' = (\mathfrak{q}, X) = (\mathfrak{n}, X) \subseteq \mathfrak{s} \subseteq \mathfrak{q}'.$$

Therefore, $\mathfrak{s} = \mathfrak{q}'$ and thus $\mathfrak{q}'$ is minimal over $(J : g)$. $\square$

Proof of Lemma 4. This lemma follows from the fact that $R_{\mathfrak{m}}$ is flat over R for every prime, and therefore every maximal, ideal $\mathfrak{m}$ and from [**11**, p. 23].

Proof of Lemma 5. Let I be an ideal in the ring R, and let a be an element of R such that $a \notin I$. Let $\mathfrak{q}$ be an element of $\operatorname{Min}(I : a)$. Let $\mathfrak{m}$ be a maximal ideal in R which contains $\mathfrak{q}$, and let $J = IR_{\mathfrak{m}}$. By assumption $\operatorname{Min}(J : a) \subseteq \operatorname{Min} J$. It must be shown that $\mathfrak{q} \in \operatorname{Min} I$.

Since $I \subseteq (I : a)$ and $(I : a) \subseteq \mathfrak{q}$, it follows that $I \subseteq \mathfrak{q}$. Assume $\mathfrak{p}$ is a prime ideal in R such that $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$.

By Lemma 4, $(J : a) = (I : a)R_{\mathfrak{m}}$.

Now let $\mathfrak{p}' = \mathfrak{p}R_{\mathfrak{m}}$ and $\mathfrak{q}' = \mathfrak{q}R_{\mathfrak{m}}$. Then, by the correspondence of ideals, respectively prime ideals, in R which are contained in $\mathfrak{m}$ and ideals, respectively prime ideals, in $R_{\mathfrak{m}}$, it follows that $\mathfrak{p}'$ is minimal over J, that $\mathfrak{q}'$ is minimal over $(J : a)$ and that $\mathfrak{p}' \subseteq \mathfrak{q}'$. So since $\operatorname{Min}(J : a) \subseteq \operatorname{Min} J$, it follows that $\mathfrak{p}' = \mathfrak{q}'$. Therefore, $\mathfrak{p} = \mathfrak{q}$ and thus $\mathfrak{q} \in \operatorname{Min} I$. $\square$

Proof of Lemma 6. Part 1 of this lemma follows from Theorem 149 [**7**, p. 108].

For part 2 it must be shown that, for any prime ideal $\mathfrak{p}$ in R, $\operatorname{ht} \mathfrak{p}' = \operatorname{ht} \mathfrak{p}$ where $\mathfrak{p}' = \mathfrak{p}R[X_1, X_2, X_3, \dots]$. For this, let $\mathfrak{p}$ be a prime ideal in R with $\operatorname{ht} \mathfrak{p} = n$, and let $\mathfrak{p}' = \mathfrak{p}R'$ where $R' = R[X_1, X_2, X_3, \dots]$.

Claim. *ht*$\mathfrak{p}' \geq n$. *Since ht*$\mathfrak{p} = n$, *there is a chain of prime ideals*

$$0 = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{p}$$

in R. This chain gives rise to the following chain of prime ideals in R'

$$0 = \mathfrak{q}_0 R' \subset \mathfrak{q}_1 R' \subset \cdots \subset \mathfrak{q}_n R' = \mathfrak{p}'.$$

This chain of prime ideals in R' implies that $\operatorname{ht} \mathfrak{p}' \geq n$, and the claim has been proven.

Claim. *$ht \mathfrak{p}' \leq n$. If this claim is not true, that is, if $ht \mathfrak{p}' > n$, then there exist prime ideals $\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_{n+1}$ in R' with*

$$0 = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n \subset \mathfrak{q}_{n+1} = \mathfrak{p}'.$$

For $1 \leq j \leq n+1$, choose $x_j \in \mathfrak{q}_j \setminus \mathfrak{q}_{j-1}$. Since $R' = \varinjlim R_i$ where $R_i = R[X_1, X_2, \dots, X_i]$ for each positive integer i, there exists a positive integer M such that $\{x_1, x_2, \dots, x_{n+1}\} \subseteq R_i$ for all $i \geq M$. Then, for all $i \geq M$, there is the following chain of prime ideals in R_i

$$0 = \mathfrak{q}_0 \cap R_i \subset \mathfrak{q}_1 \cap R_1 \subset \mathfrak{q}_2 \cap R_i \subset \cdots \subset \mathfrak{q}_n \cap R_i \subset q_{n+1} \cap R_i = \mathfrak{p}' \cap R_i.$$

Thus $\operatorname{ht}(\mathfrak{p}' \cap R_i) \geq n+1$. However, $\mathfrak{p}' \cap R_i$ is the prime ideal referred to as $\mathfrak{p}_i$ in part 1 of this lemma, so $\operatorname{ht}(\mathfrak{p}' \cap R_i) = n$ from part 1. This creates a contradiction, so the claim must be true.

Finally, putting these two claims together gives the desired result. □

Proof of Lemma 7. This follows by [**11**, p. 23] since in either case S is a flat R-module.

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