Math 30 – Fall 2009 **First Exam** Name Solutions

Instructions: You are entering the "Limit Zone!" You must use limits explicitly wherever they are called for. You do not need to quote the laws you are using, but you must use them appropriately to receive full credit. There are 80 points possible on the exam.

1. Evaluate the following limits. (8 points each)

a. $\lim_{x \to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} = \lim_{x \to 0} \frac{(\sqrt{3+x}-\sqrt{3})(\sqrt{3+x}+\sqrt{3})}{x\;(\sqrt{3+x}+\sqrt{3})}$

$$= \lim_{x \to 0} \frac{(3+x)-3}{x(\sqrt{3+x}+\sqrt{3})}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{3+x}+\sqrt{3})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{3+x}+\sqrt{3}}$$

$$= \frac{1}{\lim_{x \to 0} \sqrt{3+x}+\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}+\sqrt{3}}$$

$$= \boxed{\frac{1}{2\sqrt{3}}}$$

b. $\lim_{x \to 4} \frac{\frac{1}{4}-\frac{1}{x}}{x-4} = \lim_{x \to 4} \frac{x-4}{4x(x-4)}$

$$= \lim_{x \to 4} \frac{1}{4x}$$

$$= \frac{1}{4 \lim_{x \to 4} x} = \frac{1}{4 \cdot 4} = \boxed{\frac{1}{16}}$$

2. Find $\lim_{x\to\infty} \frac{1-x-x^2}{2x^2-7}$ Show ALL work! (8 points)

$$\lim_{x\to\infty} \frac{1-x-x^2}{2x^2-7} = \lim_{x\to\infty} \frac{1/x^2 - 1/x - 1}{2 - 7/x^2}$$

$$= \frac{\lim_{x\to\infty} \left(\frac{1}{x^2} - \frac{1}{x} - 1\right)}{\lim_{x\to\infty} \left(2 - \frac{7}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x^2} - \lim_{x\to\infty} \frac{1}{x} - 1}{2 - 7\lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{0-0-1}{2-0} = \frac{-1}{2} = \boxed{-\frac{1}{2}}$$

3. For $f(x) = \sqrt{x^2-1}$ (8 points)

a. Using the laws of continuity, explain why $f(x)$ is continuous.

$g(x) = x^2-1$ is a polynomial and therefore continuous
$h(x) = \sqrt{x}$ is a root function and therefore continuous
$f(x) = h(g(x))$ is the composition of continuous functions and therefore continuous on its domain

b. State the domain.

$$x^2 - 1 \geq 0$$
$$x^2 \geq 1$$
so $\boxed{x \leq -1 \text{ or } x \geq 1}$

(OR) $\boxed{(-\infty, -1] \cup [1, \infty)}$

c. Use the continuity of $f(x)$ to evaluate $\lim_{x\to 3} \sqrt{x^2-1}$

Since $x=3$ is in the domain of $f(x) = \sqrt{x^2-1}$,
$\lim_{x\to 3} \sqrt{x^2-1} = \sqrt{3^2-1} = \boxed{\sqrt{8} = 2\sqrt{2}}$

4. Below is the number of home runs H hit by Barry Bonds during a nine year period. Use it to estimate the instantaneous rate of change in 1998 by an appropriate method. Include the appropriate units. Show your work. (5 points)

Year	1996	1997	**1998**	1999	2000	2001	2002	2003	2004
H	42	40	**37**	34	49	73	46	45	45

Here we are thinking of $H(t)$ as a continuous variable fit to the discrete variable "home runs." Then we can estimate the instantaneous rate of change in the variable $H(t)$ in 1998 as

$$H'(1998) \cong \frac{34-40}{1999-1997} = -\frac{6}{2} = \boxed{-3} \text{ home runs per year}$$

5. Find the equation of the line tangent to the graph of $y = s(t)$ at $t = 1$ if $s(1) = 1$ and $s'(1) = -2$. (6 points)

$$y = s(1) + s'(1)(t-1)$$
$$= 1 - 2(t-1)$$
$$\boxed{y = -2t + 3}$$

6. Find c such that $f(x) = \begin{cases} \dfrac{x^2+2x-3}{x^2-1}, & x \neq 1 \\ c, & x = 1 \end{cases}$ is continuous at $x = 1$. (6 points)

To be continuous at $x = 1$, $c = \lim_{x\to 1} f(x) =$

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{(x+3)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{(x+3)}{(x+1)} = \frac{(1+3)}{(1+1)}$$

$$= \boxed{2}$$

7. The position s (in meters) as a function of time t (in seconds) of a body moving along a straight line is given by $s(t)=\frac{1}{t^2}$.

a. Find the **average** velocity of the body over the interval from $t=1$ to $t=2$. (Include the appropriate units.) (4 points)

$$v_{ave}[1,2] = \frac{s(2)-s(1)}{2-1} = \frac{\frac{1}{2^2}-\frac{1}{1^2}}{2-1} = \frac{\frac{1}{4}-1}{1}$$

$$= \boxed{-\frac{3}{4}\ m/s}$$

b. Find the **instantaneous** velocity of the body when $t=1$. (Include the appropriate units.) (8 points)

$$v(1) = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2}-\frac{1}{1^2}}{h}$$

$$= \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{1-(1+2h+h^2)}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{1-1-2h-h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2}$$

$$= \lim_{h\to 0} -\frac{\cancel{h}(2+h)}{\cancel{h}(1+h)^2}$$

$$= -\lim_{h\to 0} \frac{2+h}{(1+h)^2}$$

$$= -\frac{2+0}{(1+0)^2}$$

$$= -\frac{2}{1} = \boxed{-2\ m/s}$$

8. Using the **Squeeze Theorem**, find $\lim_{t \to \infty} e^{-t}\cos(2\pi t)$. You must show the work to receive credit.

(8 points)

$$-1 \le \cos 2\pi t \le 1$$

$$-e^{-t} \le e^{-t}\cos 2\pi t \le e^{-t}$$

$$\lim_{t\to\infty}(-e^{-t}) \le \lim_{t\to\infty} e^{-t}\cos 2\pi t \le \lim_{t\to\infty} e^{-t}$$

$$0 \le \lim_{t\to\infty} e^{-t}\cos 2\pi t \le 0$$

Therefore, $\lim_{t\to\infty} e^{-t}\cos 2\pi t = 0$ by the Squeeze Theorem

9. Use the given graph of $f(x)=\sqrt{x}$ to find a δ such that $\left|\sqrt{x}-2\right| < 0.5$ whenever $0 < |x-4| < \delta$

(5 points)

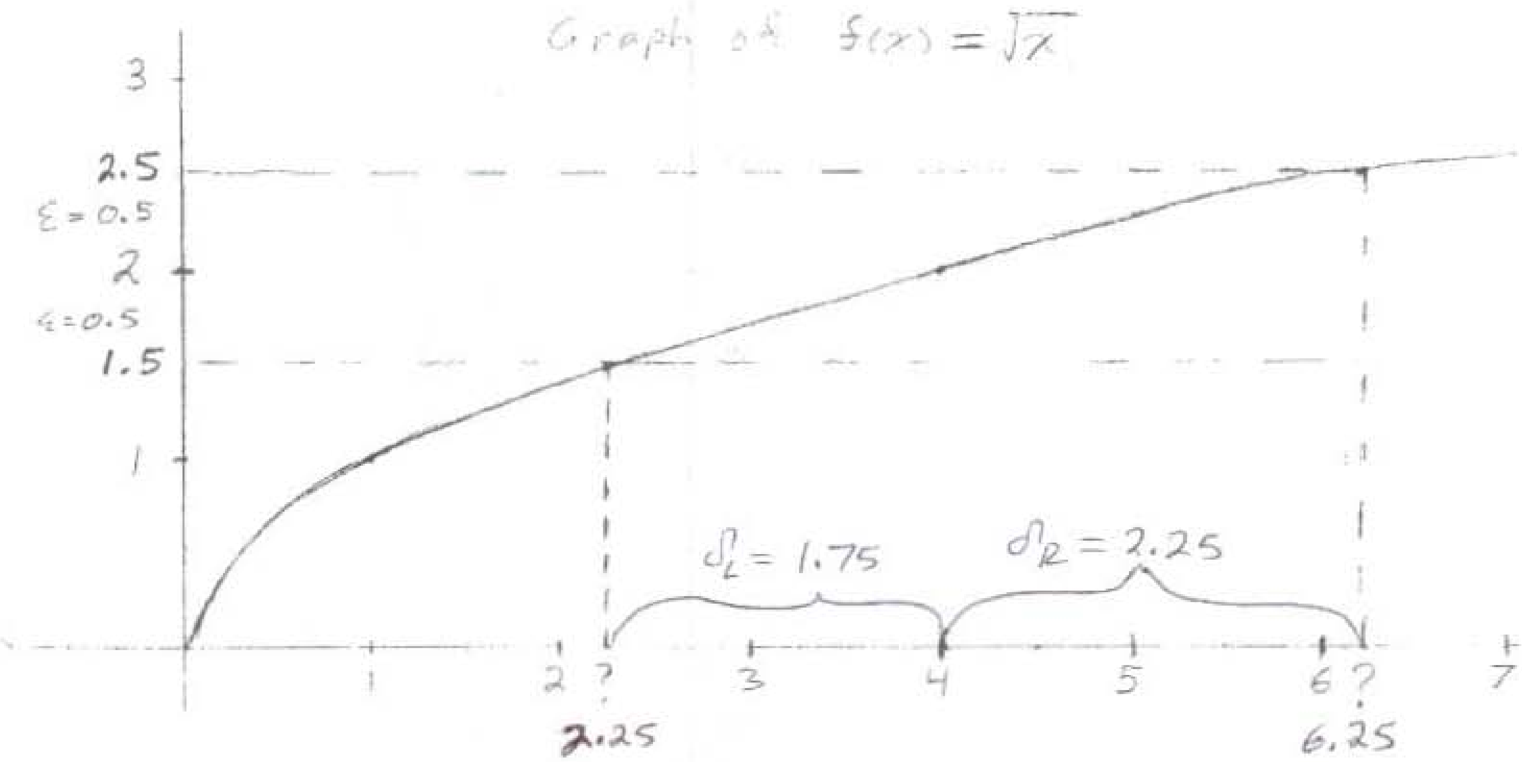
First (left-hand) $? = (1.5)^2 = 2.25$

Left-hand $\delta_L = |4 - 2.25| = 1.75$

second (right-hand) $? = (2.5)^2 = 6.25$

Right-hand $\delta_R = |4 - 6.25| = 2.25$

Let $\delta = \min\{\delta_L, \delta_R\} = \min\{1.75, 2.25\} = \boxed{1.75}$



10. On the graph paper, sketch the graph of a function that satisfies all of the given conditions. (6 points)

$\lim_{x \to 1} f(x) = \infty$ $\qquad$ $\lim_{x \to 0^-} f(x) = \infty$ $\qquad$ $\lim_{x \to 0^+} f(x) = -\infty$

$\lim_{x \to \infty} f(x) = 1$ $\qquad$ $\lim_{x \to -\infty} f(x) = 1$ $\qquad$ $\lim_{x \to \frac{1}{2}} f(x) = 0$