Math 30 – Fall 2009 **Exam II** Name Key

Instructions: No graphing calculators are allowed on the exam. Show work as appropriate. There a 80 points possible on the exam.

1. The radius of a circle is measured to be 8 cm, with a measurement error of ± 0.05 cm. (12 points)

a. Use differentials to estimate the maximum error in the calculated area of the circle. Include units.

Area $A = \pi r^2$

$dA = \left(\frac{dA}{dr}\right) dr = 2\pi r\, dr$

$\therefore \Delta A \approx dA = 2\pi(8)(0.05) = \boxed{0.8\pi}$

b. What is the relative error in the calculated area.

$$\frac{dA}{A} = \frac{2\pi r\, dr}{\pi r^2} = 2\frac{dr}{r} = 2\left(\frac{0.05}{8}\right) = \boxed{0.0125}$$

c. How accurately must the radius be measured for the percentage error in the estimated area to be less than 1%?

$$\frac{dA}{A}100\% = 2\left(\frac{dr}{r}\right)100\% = 1\%$$

so $2\left(\frac{dr}{8}\right) = 0.01$

$\boxed{dr = 0.04}$

2. Show that $y = Ce^{kt}$, where C and k are constants, satisfies the differential equation $\frac{dy}{dt} = ky$. (4 points)

$$\frac{dy}{dt} = \frac{d}{dt}Ce^{kt} = Ce^{kt} \cdot k = k(Ce^{kt}) = ky$$

3. Find the derivatives of the following functions. ***Simplify.*** (24 .s)

a. $y = 2e^{-5x}$

$y' = 2e^{-5x}(-5) = \boxed{-10e^{-5x}}$

b. $y = x^3 e^x$

$y' = 3x^2e^x + x^3e^x = \boxed{x^2e^x(3+x)}$

c. $y = \dfrac{3x-2}{2x+5}$

$y' = \dfrac{(2x+5)(3) - (3x-2)(2)}{(2x+5)^2}$

$= \dfrac{(6x+15) - (6x-4)}{(2x+5)^2} = \boxed{\dfrac{19}{(2x+5)^2}}$

d. $y = \sin(\cos x^3)$

$y' = \cos(\cos x^3)(-\sin x^3)(3x^2)$

$y' = \boxed{-3x^2(\sin x^3)\cos(\cos x^3)}$

e. $y = \tan^{-1}(\sin x)$

$y' = \frac{1}{1+\sin^2 x} \cdot \cos x = \boxed{\frac{\cos x}{1+\sin^2 x}}$

f. $y = \ln\left(x^2 \sqrt{e^x}\right) = \ln x^2 + \ln e^{\frac{1}{2}x} = 2\ln x + \frac{1}{2}x$

$y' = \boxed{\frac{2}{x} + \frac{1}{2}} = \frac{4+x}{2x}$

g. $y = \ln(\ln x)$

$y' = \frac{1}{\ln x} \cdot \frac{1}{x} = \boxed{\frac{1}{x \ln x}}$

h. $y = x^{2/5} - \pi^2$

$y' = \boxed{\frac{2}{5} x^{-3/5}}$

4. Experiments show that if the chemical reaction $N_2O_5 \to 2NO_2 + \frac{1}{2}O_2$ takes place at 113 °F, the rate of reaction of dinitrogen pentoxide is proportional to its concentration C as follows: $\frac{dC}{dt} = -0.0005C$, where the time is measured in seconds. (8 points)

a. Write the equation for the concentration C after t seconds if the initial concentration is C_0

$$C = \boxed{C_0 e^{-0.0005t}}$$

b. How long will it take to reduce the concentration to 90% of its initial value?

$$C_0 e^{-0.0005t} = 0.9C_0$$

$$-0.0005t = \ln(0.9)$$

$$t = \boxed{-2000\ln(0.9)}$$

5. For the curve curve $2xy + \pi \sin y = 2\pi$, (8 points)

a. Use implicit differentiation to find $\frac{dy}{dx}$

$$2y + 2xy' + \pi \cos y \cdot y' = 0$$

$$(2x + \pi\cos y)y' = -2y$$

$$y' = \boxed{-\frac{2y}{2x + \pi \cos y}}$$

b. Verify that the point $\left(1, \frac{\pi}{2}\right)$ is on the curve.

$$2(1)\left(\frac{\pi}{2}\right) + \pi \sin\left(\frac{\pi}{2}\right) = \pi + \pi = 2\pi$$

Problem 5 continues on the next page

c. Find the equation of the line tangent to the curve at the point $\left(1, \frac{\pi}{2}\right)$.

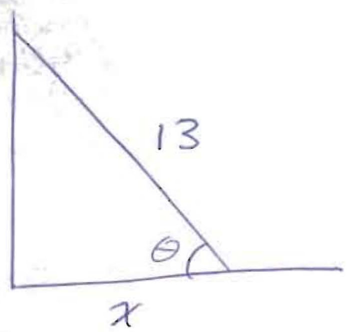
$$y' = -\frac{2\left(\frac{\pi}{2}\right)}{2(1) + \pi\cos\left(\frac{\pi}{2}\right)} = -\frac{\pi}{2}$$

$$\therefore\ y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1) \quad \text{or} \quad \boxed{y = -\frac{\pi}{2}x + \pi}$$

6. A 13-ft ladder is leaning against a wall when its base begins to slide away from the wall. When the base is 12 ft from the wall, the base is moving away from the wall at a rate of 5 ft/s. (12 points)

a. How fast is the top of the ladder sliding down the wall then? Include the appropriate units.

Let x be the distance of the base of the ladder to the wall and y be the height of the top of the ladder up the wall.



$$x^2 + y^2 = 13^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{12}{5} \cdot 5 = -12 \text{ ft/sec} \quad \text{or} \quad \boxed{12 \text{ ft/sec}}$$

b. How fast is the angle θ between the ladder and the floor changing then? Include the appropriate units.

From the picture in part (a)

$$\cos\theta = \frac{x}{13}$$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$$

$$= -\frac{1}{13\left(\frac{5}{13}\right)} \cdot 5 = -1 \quad \text{or} \quad \boxed{1 \text{ radian/sec}}$$

* Note: The use of the word "fast" in the problem indicates the answer should be positive, which is why I dropped the (−) sign in parts (a) & (b)

7. The position function of a particle is given by $s(t) = t^3 - 6t^2 + 9t + 12 \quad 0 \le t \le 3$, where s is in r[illegible] s and t is in seconds. (12 [illegible]ts)

a. Find the velocity at time t.

$v(t) = s'(t) = \boxed{3t^2 - 12t + 9 \text{ m/s}}$

b. When is the particle at rest?

$v(t) = 3(t-1)(t-3)$

so $v(t) = 0$ when $\boxed{t = 1 \text{ or } t = 3}$ seconds

c. When is the particle moving in the positive direction?

$v(t) > 0$ when $t < 1$ or $t > 3$, but on our domain (see the question), $v(t) > 0$ when $\boxed{0 \le t < 1}$

d. Find the total distance the particle travels during the three seconds.

The particle changes direction at $t = 1$, so total distance is given by $D = |s(3) - s(1)| + |s(1) - s(0)|$
$= 4 + 4 = \boxed{8 \text{ meters}}$

e. Find the acceleration at time t.

$a(t) = v'(t) = s''(t) = \boxed{6t - 12 \text{ m/s}^2}$

f. When is the particle speeding up?

The particle is speeding up when $v(t)$ and $a(t)$ have the same sign. On our domain $(0 \le t \le 3)$ this only occurs for $\boxed{1 < t < 2}$ when both are negative.