Math 30 – Fall 2009 **Exam III** Name Solutions

Instructions: No graphing calculators or cell phones are allowed on the exam. Show work as appropriate. There are 80 points possible on the exam.

1. For the graph of the function $f(x) = x^3 - 3x + 3$, (14 points)

a. On what intervals is f increasing? Decreasing?

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1) = 0$ when $x = \pm 1$

Test Points x	-2	0	2
$f'(x)$	$+$	$-$	$+$

So $f(x)$ is increasing on $(-\infty,-1) \cup (1,\infty)$

" " decreasing on $(-1,1)$

b. At what values of x does f have a local maximum? Minimum?

Critical numbers: $x = \pm 1$, from part ⓐ above

Second Derivative Test:

$f''(x) = 6x$, $f''(-1) = -6 < 0$, $f''(1) = 6 > 0$, so f has a local max at -1 and a local min at 1

c. Find the local maximum and minimum values of f.

$f(-1) = 5$

$f(1) = 1$

d. On what intervals is f concave up? Concave down?

$f''(x) = 6x$, So $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$. Therefore, $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$

e. At what values of x does f have an inflection point?

$f(x)$ changes concavity at $x = 0$

2. Find the limits. Show your work (16 points)

a. $\lim_{x \to 0^+} \frac{\ln x}{x}$ $= \boxed{-\infty}$

b. $\lim_{x \to 0} \frac{\tan 2x}{\sin 5x}$ $\overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x \to 0} \frac{2\sec^2 2x}{5\cos 5x} = \boxed{\frac{2}{5}}$

c. $\lim_{x \to \infty} (x \ln x - 5x)$ $= \lim_{x \to \infty} x(\ln x - 5) = \boxed{\infty}$

d. $\lim_{x \to 0^+} x^x$ $= \lim_{x \to 0^+} e^{\ln x^x} = \lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x}$

$= e^{\lim_{x \to 0^+} \left(\frac{\ln x}{1/x}\right)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} e^{\lim_{x \to 0^+} \frac{1/x}{-1/x^2}} = e^{\lim_{x \to 0^+} -x}$

$= e^0 = \boxed{1}$

3. Find the absolute maxima and absolute minima for f on the given interval. (16 points)

a. $f(x)=2x^3-3x^2-12x+1$ on [-2, 3]

$f'(x) = 6x^2-6x-12 = 6(x^2-x-2) = 6(x+1)(x-2)$

The Extreme Value Theorem Applies, so

x	-2	-1	2	3
$f(x)$	-3	8	-19	-8

Absolute maximum: $(-1, 8)$

Absolute minimum: $(2, -19)$

b. $f(x)=\dfrac{x}{x^2+1}$ on [0, 2]

$f'(x) = \dfrac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

The Extreme Value Theorem Applies on $[0,2]$, so

x	0	1	2
$f(x)$	0	$1/2$	$2/5$

Absolute maximum: $(1, \frac{1}{2})$

Absolute minimum: $(0, 0)$

4. Verify that the function $f(x)=x^3+x-1$ satisfies the hypotheses of the Mean Value Theorem on the interval [0, 3], (make sure to state what the hypotheses are), then find all numbers c that satisfy the conclusions of the Mean Value Theorem. (8 points)

$f(x)=x^3+x-1$ is a polynomial, so it is continuous on $[0,3]$ and differentiable on $(0,3)$. Then by the Mean Value Theorem, there is a c in $(0,3)$ such that $f'(c) = \frac{f(3)-f(0)}{3-0} = \frac{29-(-1)}{3} = \frac{30}{3} = 10$

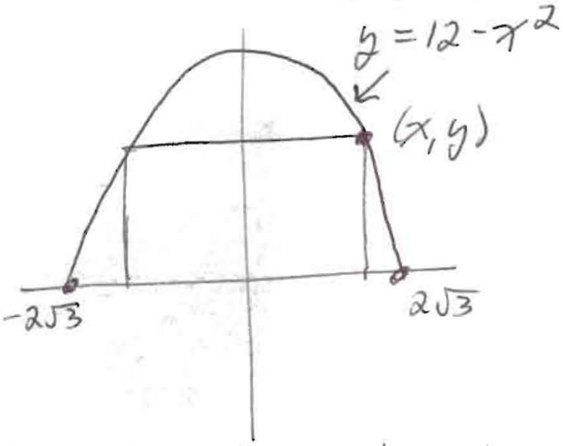
Setting $f'(c) = 3c^2+1 = 10$,

$c^2 = 3$

$c = \pm\sqrt{3}$, but only

$\boxed{c=\sqrt{3}}$ is in $(0,3)$

5. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions? (10 points)



Area to be maximized,

$A = 2xy$ subject to the constraint $y = 12 - x^2$, so

$A(x) = 2x(12-x^2) = 24x - 2x^3$,

for $0 \le x \le 2\sqrt{3}$

$A'(x) = 24 - 6x^2 \equiv 0$, so the only critical number in $(0, 2\sqrt{3})$ is $x = 2$. By the Extreme Value Theorem,

x	0	2	$2\sqrt{3}$
$A(x)$	0	32	0

So the maximum area is $\boxed{32}$ and the dimensions of the rectangle is $\boxed{4 \times 8}$

6. Find the following functions. (16 points)

a. The most general antiderivative of the function $f(\theta) = \cos\theta - 4\sin\theta$. (Check your answer by differentiation.)

$$\boxed{F(\theta) = \sin\theta + 4\cos\theta + C}$$

Check, $F'(\theta) = \cos\theta - 4\sin\theta = f(\theta)$

b. Find f given $f''(x) = 12x^2 - 6x + 2,\quad f'(0) = 5,\quad f(0) = -1$

$f'(x) = 12\frac{x^3}{3} - 6\frac{x^2}{2} + 2x + C = 4x^3 - 3x^2 + 2x + C$

and $f'(0) = C = 5$

$\therefore f'(x) = 4x^3 - 3x^2 + 2x + 5$

$f(x) = 4\frac{x^4}{4} - 3\frac{x^3}{3} + 2\frac{x^2}{2} + 5x + D$

$= x^4 - x^3 + x^2 + 5x + D$

and $f(0) = D = -1$

$\therefore \boxed{f(x) = x^4 - x^3 + x^2 + 5x - 1}$