1. (15 pts) Prove the following statements. Use only the field and order axioms on the last page of this test and the fact that $x \cdot 0 = 0$ for all $x \in \mathbb{R}$. Justify each step.

(a) If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.

Since $z \neq 0$, $\frac{1}{z}$ exists as a real number by M5.

$$(x \cdot z) \cdot \frac{1}{z} = (y \cdot z) \cdot \frac{1}{z} \quad \text{by M1}$$

$$x \cdot (z \cdot \frac{1}{z}) = y \cdot (z \cdot \frac{1}{z}) \quad \text{by M3}$$

$$x \cdot 1 = y \cdot 1 \quad \text{by M5}$$

$$x = y \quad \text{by M4}$$

(b) $-1 \cdot x = -x$.

$$x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x \quad \text{by M4 and M2}$$

$$= x \cdot 1 + x \cdot (-1) \quad \text{by M2}$$

$$= x (1 + (-1)) \quad \text{by DL}$$

$$= x \cdot 0 \quad \text{by A5}$$

$$= 0 \quad \text{since } x \cdot 0 = 0 \forall x \in \mathbb{R}.$$  

$$\therefore x + (-1) \cdot x = 0.$$  

This means $-1 \cdot x$ is the additive inverse of $x$. Since the additive inverse is unique, $-x = -1 \cdot x$.

(c) If $a < b$ and $c < d$, then $a + c < b + d$.

* $a + c < b + c \quad \text{by C3}$

and $c < d \Rightarrow c + b < d + b \quad \text{by C3}$

$$\Rightarrow b + c < b + d \quad \text{by A2} \quad \star \star$$

(By * and ** and C2, $a + c < b + d$.)

Name: Solutions

Directions: There are 100 points possible. Please cross out any scratch work but marking an x over it, but do not obliterate it. Take a deep breath, show your work and do well!
2. (26 pts) Find the supremum and infimum of each set. (No proof needed except for part (e).)

(a) \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \)
\[
\inf S = 0 \\
\sup S = 1
\]

(b) \( S = \{ \frac{1}{n} : n \in \mathbb{Z}, n \neq 0 \} \)
\[
\sup S = 1 \\
\inf S = -1
\]

(c) \( S = \{ \frac{2}{n} : n \in \mathbb{N}, n \neq 1 \} \)
\[
\sup S = 2 \\
\inf S = 1
\]

(d) \( S = \{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \} \)
\[
\text{If } n \text{ even, } \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \\
\text{If } n \text{ odd, } 0, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, \ldots
\]
\[
\sup S = 1/2 = 3/2 \\
\inf S = -1
\]

(e) Assuming the infimum of a set exists, prove that it is unique. Let \( S \) be a set.
Assume \( l \) and \( l' \) are both infima of \( S \). Regarding \( l \) as a lower bound and \( l' \) as a greatest lower bound, we have that
\[
l \leq l' \\
(\star)
\]
Regarding \( l' \) as a lower bound and \( l \) as a greatest lower bound,
\[
l' \leq l \\
(\star \star)
\]
Combining \( \star \) and \( \star \star \), \( l = l' \).
3. (15 pts) Let $S = \{2 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove $\sup S = 2$.

(1) Claim: $2$ is an upper bound (UB) of $S$

Proof: Clearly, $\frac{1}{n} > 0 \Rightarrow 0 > -\frac{1}{n}$

$\Rightarrow 2 > 2 - \frac{1}{n}$ \hspace{1cm} \forall n \in \mathbb{N}$

$\therefore 2$ is an UB of $S$

(2) Claim: $2$ is the least upper bound of $S$

Proof: Let $m' < \sup S = 2$ (to show $m'$ is not an UB of $S$)

By the Archimedean Principle,

$\exists n \in \mathbb{N} \Rightarrow n > \frac{1}{2-m'}$

$\Rightarrow 2 - m' > \frac{1}{n}$ since $2 - m' > 0$

$\Rightarrow 2 - \frac{1}{n} > m'$

Since $2 - \frac{1}{n} \in S$, $m'$ is not an UB of $S$

$\forall m' < 2$. \therefore 2 is the least UB of $S$

$\therefore 2 = \sup S$
4. (15 pts)

5 (a) State the Completeness Axiom.

Every nonempty subset of the real numbers that is bounded above has a supremum.

5 (b) Use the Completeness Axiom to prove the Archimedean Property which states (in one form) “The natural numbers are not bounded above by any real number.”

By contradiction, suppose the natural numbers, \( N \), are bounded above by a real number, say \( r \). By the Completeness Axiom, \( \sup(N) \) exists, let \( m = \sup(N) \).

Then, \( \exists n \in N \) \( \Rightarrow m - 1 < n \leq m \).

But \( m - 1 < n \Rightarrow m < n + 1 \) and \( n + 1 \in N \)

\( \Rightarrow \) since \( m = \sup(N) \), no natural number can exceed \( m \).

\( \therefore \) the natural numbers are not bounded above by any real number.

5 (c) Prove that if \( x > 0 \) and \( y \in \mathbb{R} \), then there exists \( n \in \mathbb{N} \) such that \( nx > y \). Hint: Use some algebra and the Archimedean Property.

By the Archimedean Property, \( \exists n \in \mathbb{N} \) \( \Rightarrow \)

\[
\frac{y}{x} > \frac{y}{x} \text{ since } \frac{y}{x} \in \mathbb{R}.
\]

Since \( x > 0 \), \( n \cdot x > \left(\frac{y}{x}\right)x \)

\( \Rightarrow \) \( nx > y \)
5. (20 pts) Using only the definition of convergence of a sequence, prove

\[ \lim_{n \to \infty} \frac{2n+5}{3n+1} = \frac{2}{3} \]

Let \( \varepsilon > 0 \). Let \( N = \frac{13}{9\varepsilon} \). For all \( n > N \),

\[ \left| \frac{2n+5}{3n+1} - \frac{2}{3} \right| = \left| \frac{3(2n+5) - 2(3n+1)}{3(3n+1)} \right| \]

\[ = \left| \frac{13}{9n+3} \right| = \frac{13}{9n+3} < \frac{13}{9n} \leq \frac{13}{9N} = \frac{13}{9 \left( \frac{13}{9\varepsilon} \right)} = \varepsilon \]

\[ \therefore \lim_{n \to \infty} \frac{2n+5}{3n+1} = \frac{2}{3} \]
6. (15 pts)

(a) Prove \( x \leq |x| \) for all \( x \in \mathbb{R} \).

Case I: \( x \geq 0 \)

\[ |x| = x \geq x \]
\[ \therefore |x| \geq x \]

Case II: \( x < 0 \)

Then \( |x| = -x \)

Since \( x < 0 \), \( 0 < -x \).

\[ \Rightarrow x < 0 < -x \quad \text{combining the last two inequalities} \]
\[ \Rightarrow x < 0 < -x = |x| \]
\[ \Rightarrow x < |x| \Rightarrow x \leq |x| \]

(b) Prove \( |x| - |y| \leq |x - y| \). (Hint: Use the Reverse Triangle Inequality. If you have forgotten it, I will write it down for you, but it will cost you 3 points out of the 5 points possible for this part.)

By the Reverse Triangle Inequality,

\[ |x| - |y| \leq |x - y| \quad \ast \]

By part (a), \( |x| - |y| = |x| - |y| \) \quad \ast \ast

Combining \( \ast \) and \( \ast \ast \),

\[ |x| - |y| \leq |x| - |y| \leq |x - y| \]

(c) Prove: if \( |x - y| < c \), then \( |x| < |y| + c \).

By part (b),

\[ |x| - |y| \leq |x - y| \]

Since \( |x - y| < c \), transitivity gives

\[ |x| - |y| < c \]
\[ \Rightarrow |x| < |y| + c \]