Let $s_1 = 4$ and define $s_{n+1} = \sqrt{4s_n - 1}$ for $n$ in the natural numbers. Prove that $\{s_n\}$ is decreasing and bounded. Find $\lim s_n$.

Claim: $\{s_n\}$ is decreasing.

Proof by induction on $n$. Let $P(n)$ be $s_n \geq s_{n+1}$.

(i) Is $P(1)$ true? $s_1 \geq s_2$?

\[
s_1 = 4, \quad s_2 = \sqrt{4s_1 - 1} = \sqrt{4(4) - 1} = \sqrt{15} \leq 4 = s_1.
\]

\[s_2 \leq s_1.\]

(ii) Assume $P(n)$ is true, i.e., $s_n \geq s_{n+1}$. Show $P(n+1)$ is true, i.e., $s_{n+1} \geq s_{n+2}$.

Since $s_n \geq s_{n+1}$, we have $4s_n \geq 4s_{n+1}$ (over)

\[
\Rightarrow \sqrt{4s_n - 1} \geq \sqrt{4s_{n+1} - 1} \Rightarrow s_{n+1} \geq s_{n+2}
\]

Quiz 1 Replacement Question: Do only if you want to replace your score from Quiz 1.

Using only the definition of divergence to $+\infty$. Prove $\{3n^2 + 1\}$ diverges to $+\infty$.

Let $M \in \mathbb{R}$. Let $N = M$. For all $n > N$,

\[3n^2 + 1 > 3N^2 + 1 > 3N^2 > N^2 > N = M\]

\[\therefore 3n^2 + 1 > M \quad \forall n > N\]

\[\therefore \lim 3n^2 + 1 = +\infty\]
Claim: \( \{a_n^3\} \) is bounded below by 1.

By induction. Let \( P(n) \) be the statement
\[ a_n \geq 1. \]

i) \( P(1) \) is true since \( a_1 = 4 \geq 1 \)

ii) Assume \( P(n) \) is true, i.e. \( a_n \geq 1 \). Show
\( P(n+1) \) is true, i.e. \( a_{n+1} \geq 1 \).

\[ a_n \geq 1 \Rightarrow 4a_n \geq 4 \]
\[ \Rightarrow 4a_n-1 \geq 3 \]
\[ \Rightarrow \sqrt{4a_n-1} \geq \sqrt{3} \]
\[ \Rightarrow a_{n+1} \geq \sqrt{3} > 1 \]
\[ \therefore a_{n+1} \geq 1 \]

Since \( \{a_n^3\} \) is decreasing, it is bounded above by \( a_1 = 4 \). Thus, \( \{a_n^3\} \) is bounded and monotone, and therefore, converges to some \( a \in \mathbb{R} \).

Since \( \lim a_{n+1} = \lim a_n = a \), we have
\[ a_{n+1} = \sqrt{4a_n-1} \]
\[ \Rightarrow a = \sqrt{4a-1} \]
\[ \Rightarrow a^2 = 4a - 1 \]
\[ \Rightarrow a^2 - 4a + 1 = -1 \]
\[ \Rightarrow a^2 - 4a + 4 = 3 \]
\[ \Rightarrow (a-2)^2 = 3 \Rightarrow a - 2 = \pm \sqrt{3} \]
\[ \Rightarrow a = 2 \pm \sqrt{3} \]

but \( a_n \geq 1 \) so
\[ \lim a_n = 2 + \sqrt{3} \approx 2.73 \neq 1 \]