

SPRING 2014 : STAT - 1 , Section 10.

HANDOUT - CHAPTER 4 - PROBABILITY

- We study PROBABILITY CONCEPTS in this chapter. You should understand the notions of *Sample Space*, *Outcome*, *Event*, *Mutually Exclusive Events* (p.157), *Probability*, *Equally Likely Outcomes*, *Frequentist interpretation of probability* and basic properties of Probability. We will use Example 4.3 (p.147 – two balanced dice), Example 4.5 (p.153 – standard deck of cards), Tossing a coin once, Tossing a coin twice and Tossing a coin thrice often in this chapter.

TRY exercises 4.1, 4.3, 4.4, 4.7, 4.9, 4.15.

- For two sets A, B , their union A or B (at least one of A, B occur), their intersection A & B (both A, B must occur), complement of A denoted by *not* A (A does not occur) are related events. Use Venn diagrams to understand these relations. (see Fig. 4.9, p.154) We then have (i) $P(S) = 1$ (S is a sure event) (ii) $P(\phi) = 0$ (ϕ is the impossible event) (iii) Special Addition Rule (formula 4.1, p.162), (iv) The General Addition Rule (Formula 4.3, p.164) and (v) The Complement Rule (Formula 4.2, p.163). Go over Examples 4.10 , 4.11, 4.12, 4.13.

TRY Exercises 4.40, 4.43, 4.47, 4.49, 4.69, 4.75, 4.79, 4.80, 4.81, 4.82, 4.83.

- Contingency tables or two-way tables represent frequency distribution of two qualitative variables. In section 4.4, study Example 4.14, 4.15 (p. 168–169), 4.17 (p.175) to learn *joint probabilities* , *marginal probabilities*, *joint probability distribution* and *conditional probabilities*. $P(B|A)$ is the conditional probability of B given A and is defined by $P(B|A) = \frac{P(A \& B)}{P(A)}$ whenever $P(A) \neq 0$. Learn the *General Multiplication Rule* (p.181), *Independent Events* (p.183), *Dependent Events*, the *Special Multiplication Rule for Independent Events* (p.184) and how to check for independence. Study pages 180–185.

TRY 4.88, 4.91, 4.95, 4.105, 4.107, 4.111, 4.113, 4.118, 4.120, Correlation of events (p.180), 4.121, 4.124, 4.133, 4.135, 4.136.

- The following is an outline and derivation of the celebrated *Bayes' Rule* in Section 4.74. Using the discussion you should be able to solve 4.157, 4.161, 4.164.
- Reverend Thomas Bayes, indulging in mathematical pursuits in the second half of the 18th century, discovered a formula in probability called the Bayes' Rule. It is widely applied since the second half of the 20th century leading to a new area called *Bayesian inference* in statistical analysis of data.

Consider a collection A_1, A_2, \dots, A_k of k mutually exclusive events which are also exhaustive (A_1 or A_2 or \dots ; or $A_k = S$); such a collection is called a collection of *hypotheses* since one and exactly one of them is true under the random experiment leading to the sample space under consideration. Now let B be an event that we have observed (*evidence*). Suppose we know the prior probabilities of the hypotheses

$$P(A_1), P(A_2), P(A_3), \dots, P(A_k) \text{ and}$$

the conditional probabilities of the evidence under the various hypotheses

$$P(B|A_1), P(B|A_2), P(B|A_3), \dots, P(B|A_k)$$

then the Bayes formula derives the posterior probabilities of the hypotheses incorporating the evidence in two steps:

$$\text{STEP 1 (TOTAL PROBABILITY)} : P(B) = \sum_{j=1}^k P(A_j) \times P(B|A_j)$$

$$\begin{aligned} \text{Proof: } P(B) &= P((B \& A_1) \& (B \& A_2) \& \dots \& (B \& A_k)) \\ &= P(A_1) \times P(B|A_1) + P(A_2) \times P(B|A_2) + \dots + P(A_k) \times P(B|A_k) \\ &\text{(use multiplication rule and mutually exclusiveness of the hypotheses)} \\ &= \sum_{j=1}^k P(A_j) \times P(B|A_j). \end{aligned}$$

$$\text{STEP 2 : (BAYES' RULE)} \quad P(A_i|B) = \frac{P(A_i) \times P(B|A_i)}{\sum_{j=1}^k P(A_j) \times P(B|A_j)}$$

for any $i = 1, 2, \dots, k$ since

$$P(A_i|B) = \frac{P(A_i \& B)}{P(B)} = \frac{P(A_i) \times P(B|A_i)}{P(B)}$$

and using STEP 1 result.

Example: Suppose we have three identical wallets named I, II, III ; Wallet I has ten \$10 bills and five \$1 bills, Wallet II has four \$10 bills and eleven \$1 bills, and Wallet III has fifteen \$1 bills. A wallet is drawn at random. Let A_1 = Wallet I is drawn, A_2 = Wallet II is drawn, and let A_3 = Wallet III is drawn. Then $P(A_1) = P(A_2) = P(A_3) = (1/3)$ are the prior probabilities. Let a bill be selected at random from the drawn wallet and let B = the event that the selected bill is a \$10 bill. We can compute

$$P(B|A_1) = (10/15), P(B|A_2) = (4/15), P(B|A_3) = (0/15).$$

Applying STEP 1 we get

$$P(B) = (1/3) \times (10/15) + (1/3) \times (4/15) + (1/3) \times (0/15) = (14/45)$$

and

$$P(A_1|B) = \frac{(1/3) \times (10/15)}{(14/45)} = (10/14)$$

$$P(A_2|B) = \frac{(1/3) \times (4/15)}{(14/45)} = (4/14)$$

$$P(A_3|B) = \frac{(1/3) \times (0/15)}{(14/45)} = (0/14) = 0.$$

The next sheet contains two problems based on Bayes' Rule. Solve these and we will discuss them later in class.

NAME:

- The probability that a student studies for a test (recall Midterm II is on Thursday, 4/3/14) is 0.7. Given that she studies, the probability that she will pass is 0.8. Given that she does not study, the probability is 0.15 that she will pass.

(a) Define the following EVENTS suitably:

$$A_1 =$$

$$A_2 =$$

$$B =$$

(b) Find the probability that she will pass the test.

(c) Find the probability that she did not study given that she passed the course.

- Customers who purchase a certain make of car can order an engine in any of three sizes. Of all cars sold, 50% have the small engine, 30% have medium-size engine, and 20% have the large engine. Of cars with the small engine, 5% fail an emissions test within two years, while 15% of those with medium-size engine and 10% with large engine fail the test within two years.

(a) Define events A_1, A_2, A_3 and B suitably.

(b) What is the probability that a randomly chosen car of that make will fail an emissions test within two years?

(c) A randomly chosen car of that make failed the emission test within two years. What is the probability that it is a car with medium size engine?