Chapter 2

Infinite-Horizon and Overlapping-Generations Models

This chapter uses the same assumptions as the Solow Growth model, with one key difference: the savings rate is endogenously determined by utility-maximizing consumers. Therefore, the evolution of the capital stock is determined in a general equilibrium model with households maximizing lifetime utility and perfectly-competitive firms maximizing profits. In the Solow Growth Model, the supply of capital stock is essentially fixed because the savings rate is exogenous and constant. Here, households supply capital to firms in order to maximize utility. Changes in the model's parameters (population growth rate, depreciation rate, production function, etc.) affect both the supply and demand of capital. There are some key terms one should be familiar with when reading this chapter:

- **Euler equation** (AKA Euler condition). A first-order condition that demonstrates a tradeoff of a variable across time or states of the world. In macroeconomics, the most common Euler equations will show the tradeoff between consumption today and consumption in the future.

- **Pareto efficiency.** In the discussion of this model, one will often see reference to a concept of Pareto efficiency. A Pareto improvement refers to moving to a different outcome such that one individual is better off without making anyone else worse off. Pareto efficiency means that no further Pareto improvements are possible. This is a formal way of defining efficiency used by economists. Less formally, Pareto efficiency means there is no waste.

- **Representative household** (AKA representative consumer). This refers to a household used to formalize the utility choices of consumers. Of course, all individuals make their own choices, but in modeling their behavior as a group, it is useful to simplify things to assume that all households are the same, with their decisions being represented by the representative household. Macroeconomic models involve making abstract choices about generic baskets of goods (e.g., consumption), so we're assuming that individual differences about consumption are at least similar across households. It isn't that macroeconomists believe that households don't have different utility functions when it comes to consumption bundles, it is more that they believe these differences are not critical to outcomes.

- **Representative firm.** Like the representative household, this representative firm is designed to represent the profit-maximizing choices of all firms. This is far less controversial than using a representative household – firms clearly have the same objective whereas households may be irrational with their objectives not clearly defined.

- **Social planner.** The social planner is the one who insures that the economy reaches Pareto efficient outcomes. In a perfectly competitive world, the market outcomes are usually Pareto efficient (e.g., the same ones the social planner would choose). Discussions of the social planner arise when discussing the concept of social welfare where market outcomes are not Pareto efficient.

- **Social welfare.** Social welfare refers to the collective benefits to households and firms. A more familiar name for this is total surplus (consumer surplus + producer surplus).

Part B The Diamond Model

The Diamond model is an overlapping-generations (OLG) model defined in discrete time. The Solow Growth model and the Ramsey-Cass-Koopmans model (from Part A) are cast in continuous time with households and firms who "live" forever. The Diamond model assumes that in each period, there are households entering the workforce and households exiting (e.g., retirement). This approach to modeling has some drawbacks. Most important is the difficulty in evaluating social welfare in OLG models. The model implies that the choice of time period matters (e.g., whether a household is working or retired)

2.1 Assumptions

- *Discrete time, two-period model.* The households live for two periods: (1) working and (2) retirement. After retirement, the household dies.
- *Population growth rate.* Population grows at rate $n : L_t = (1+n)L_{t-1}$. This implies that in any period t there are L_t individuals born (working) and L_{t-1} individuals in retirement.
- *Labor supply and lifetime income.* Each household supplies one unit of labor in period (1), earning income $= A_t w_t \times 1$. The worker earns the real wage w_t and the benefit of labor-augmenting technological progress, A_t, from the one unit of labor supplied. The lifetime income is divided between the two periods of life to pay for consumption in each period.
- *Savings.* The household spends a portion of lifetime income in period (1) on consumption , C_{1t}. The remainder $(A_t w_t - C_{1t})$ is saved to pay for consumption in period (2) , C_{2t+1}. The savings earn interest $(1 + r_{t+1})$ on each unit of output saved. Since households die at the end of period (2), $C_{2t+1} = (1 + r_{t+1})(A_t w_t - C_{1t})$.
- *Lifetime utility.* Households choose consumption each period, C_{1t} and C_{2t+1}, to maximize lifetime utility:

  $$U_t = U\left(C_{1t}, \frac{1}{1+\rho} C_{2t+1}\right)$$

  with $\rho > 0$. The term $\frac{1}{1+\rho}$ is known as the discount factor. This is a common way to model lifetime consumption choices. All else equal, households value consumption tomorrow relatively less than consumption today. Mathematically, this assumption is necessary to make the model tractable. For example, if households valued consumption equally in each period, we would have no way to identify how much they will consume in each period. Romer assumes the constant relative risk aversion (CRRA) utility function:

  $$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

with $\theta > 0$. The CRRA utility function has properties that are appealing to researchers in macroeconomics and finance. Specifically, this utility function assumes that the coefficient of relative risk aversion is equal to θ and therefore it is independent of the households consumption choices. In other words, no matter how much or how little the household consumes, its aversion to risk is the same. This property is necessary for the model to generate a balanced growth path. It can be shown that as $\theta \to 0$, the CRRA utility function approaches a log utility function:

$$U_t = \log(C_{1t}) + \frac{1}{1+\rho}\log(C_{2t+1})$$

- *Production.* Firms choose capital K_t and labor L_t to maximize profits according to the following production function:

  $$Y = F\left(K_t, A_t L_t\right)$$

  The profit maximizing problem yields the following:

  $$\begin{aligned} r_t &= f'(k_t) \\ w_t &= \left[f(k_t) - k_t f'(k_t)\right] \end{aligned}$$

  Note, in Romer, the variables k_t, y_t, c_t, etc. are defined in units of effective labor Jones defines these use the tilde: $\widetilde{k}, \widetilde{y}$, and so on. This guide will use the Romer notation to maintain consistency with the chapter. Note that the worker earns w_t for each unit of labor L_t supplied. Each effective worker earns $A_t w_t$ for each unit.

- *Technological progress.* Technology grows at rate $g : A_t = (1+g)A_{t-1}$.

2.2 Household Behavior

The household's lifetime budget constraint is given by:

$$C_{1t} + \frac{1}{1+r_{t+1}}C_{2t+1} = A_t w_t$$

The household's lifetime income must therefore be equal to the household's present value of lifetime consumption. Period (2) consumption, C_{2t+1} is "discounted" to the present.

The household's maximization problem can be expressed as:

$$L = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda_t \left(A_t w_t - C_{1t} - \frac{1}{1+r_{t+1}} C_{2t+1} \right)$$

This yields the following first-order conditions:

$$\begin{aligned} C_{1t} &: \quad C_{1t}^{-\theta} - \lambda_t = 0 \\ C_{2t+1} &: \quad \frac{1}{1+\rho} C_{2t+1}^{-\theta} - \lambda_t \frac{1}{1+r_{t+1}} = 0 \end{aligned}$$

Combining and rewriting, we find the following Euler equation:

$$C_{1t}^{-\theta} = \frac{1+r_{t+1}}{1+\rho} C_{2t+1}^{-\theta}$$

The term on the left is the marginal utility of consumption today. The term on the right is the marginal utility of consumption tomorrow, discounted to the present. We can substitute out for C_{2t+1} (using the definition given above) to solve for C_{1t} in terms of the parameters and lifetime income:

$$C_{1t} = \underbrace{\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}}_{\text{savings rate } s(r_{t+1})} A_t w_t$$

The utility-maximizing household defines the savings rate as a function of the discount rate and the interest rate. The savings rate is increasing in r_{t+1} - an increase in the interest rate causes households to save more, and therefore increase second-period consumption. Using the lifetime budget constraint and the solution for period (1) consumption above, we can solve for C_{2t+1} :

$$C_{2t+1} = (1+r_{t+1})\,(1 - s(r_{t+1})) A_t w_t$$

The savings rate is:

$$s(r_{t+1}) = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

The size of the parameter θ has important implications for how households respond to changes in the interest rate. This is because the parameter θ measures the household's willingness to substitute consumption across time. When making a choice about consumption today versus tomorrow (e.g., consuming versus saving today), one must consider the substitution and income effects. Suppose that the interest rate increases - what is the effect on savings?

- Substitution effect. The higher interest rate means that C_{1t} is relatively more expensive than C_{2t+1}, causing households to substitute toward future consumption. The result is an increase in savings.

- Income effect. The increase in the interest rate implies the household's earnings from capital investment are higher, increasing the lifetime earnings available for consumption. This should cause the household to save less of its income, increasing C_{1t}.

It can be shown that when $\theta < 1$, the substitution effect dominates. That is, the benefits of an increase in the interest rate cause the household to save more and consume less today. When $\theta > 1$, the income effect dominates. In this case, the households will respond to an increase in the interest rate by consuming more today. To demonstrate this mathematically, differentiate the savings function with respect to r. The sign of the derivative depends on the value of θ.

2.2.1 Special case: Log Utility $\theta = 1$

The following sections, Romer makes use of the log utility function to solve for the balanced growth path. Here, we derive the savings rate under these assumptions. To solve for the log utility case, the household's consumption choice is defined by the following Lagrangian:

$$L = \log(C_{1t}) + \frac{1}{1+\rho}\log(C_{2t+1}) + \lambda_t \left(A_t w_t - C_{1t} - \frac{1}{1+r_{t+1}} C_{2t+1} \right)$$

The Euler equation is found by combining the first-order conditions:

$$\frac{1}{C_{1t}} = \frac{1+r_{t+1}}{1+\rho}\frac{1}{C_{2t+1}}$$

Rewriting this expression and substituting out for C_{2t+1} :

$$C_{1t} = \frac{1+\rho}{2+\rho} A_t w_t$$

Therefore, the savings rate is independent of the interest rate in this case. The savings rate, $s = \frac{1}{2+\rho}$. When $\theta = 1$ the substitution and income effects associated with a change in the interest rate are offset by one another, making the savings rate independent of the interest rate.

2.3 The Dynamics of the Economy

The law of motion for the capital stock is defined by how much households save. In per effective worker terms, the law of motion is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t$$

Note, $w_t = f(k_t) - k_t f'(k)$ is functions of the amount of capital per effective worker purchased today t. The savings rate is a function of $r_{t+1} = f'(k)$, which is a function of the capital stock next period, $t+1$. We can therefore express the capital stock next period in terms of the model parameters and the capital stock today k_t :

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k)) \left[f(k_t) - k_t f'(k) \right]$$

Like the Solow model, the balanced growth path occurs when the capital stock per effective worker is not changing. That is, when $k_{t+1} = k_t$, so that $\Delta k = 0$. We cannot go further with the expression above. While the model does have an implicit solution from the expression above, it does not have a closed-form solution. We are able to show that the capital stock will converge to a steady state value, but we cannot solve for this value explicitly.

2.3.1 Logarithmic Utility and Cobb-Douglas Production

However, if we assume a Cobb-Douglas production function and log utility, we can solve for the steady state level of capital per effective worker. From above, the savings rate is constant in the log utility case:

$$s = \frac{1}{2+\rho}$$

The real wage rate per effective worker is:

$$A_t w_t = (1-\alpha) k_t^{\alpha}$$

Therefore, the law of motion for the capital stock is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha}$$

Since the Diamond model is a two-period model, it doesn't have a diagram analogous to the Solow Growth Model. We can use the expression above to see how a change in the model parameters affect outcomes. The underlying dynamics and convergence to steady state is similar to Solow. Consider the following:

- Increase in population growth rate n :

  $$k_{t+1} > \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha) k_t^\alpha \Longrightarrow \Delta k > 0 \Rightarrow k \uparrow \text{ until } \; k_{t+1} = k_t = k^*_{new}$$

- Increase in savings rate (decrease in ρ) :

  $$k_{t+1} < \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha) k_t^\alpha \Longrightarrow \Delta k < 0 \Rightarrow k \downarrow \text{ until } \; k_{t+1} = k_t = k^*_{new}$$

  We maintain the same basic implications as the Solow Growth model. The fundamental difference in the Diamond model is that the savings rate is determined by households maximizing utility. The key implications for economic growth are identical:

- The growth rates of key variables are identical. Specifically, per capita output grows at rate g.

- Changes to the model parameters (besides g) lead to changes in steady state, but do not lead to changes in the growth rate of variables in per capita terms. In other wards, a change in the savings rate affects per capita income, but does not affect its growth rate.

2.3.2 The Speed of Convergence

At steady state, $k_{t+1} = k_t = k^*$:

$$k^* = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha) k^{*\alpha}$$

$$k^* = \left[\frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

Solving for y^* :

$$y^* = \left[\frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$$

The speed of convergence to steady state depends on capital share of output α. If there is a change in the model's parameters, capital per effective worker gets $(1-\alpha)$ of the way to the new steady state value each period. This makes sense because the transition to a new steady state is based on the accumulation/decumulation of capital per effective worker. If α is low, it will take relatively longer for this process to occur.

For a given value of α, the economy will converge to steady state more quickly in the Diamond model vs. the Solow model.

2.3.3 General Case

In the general case (CRRA utility and a generic production function $f(k)$), Romer shows that a wide variety of outcomes are possible. That is, the steady state in the model can lead to situations where k^* is unstable in the sense that there are multiple equilibria possible, or that k collapse to zero or explodes (continually grows). The possibility shown in Panel (d) of the textbook is an interesting one that is becoming more popular in research. The existence of multiple equilibria suggests that the actual outcome is arbitrary (among the multiple outcomes that are consistent with utility maximization). This means it is possible for a sudden change in expectations affect actual outcomes.

2.4 The Possibility of Dynamic Inefficiency

OLG models are difficult to evaluate in terms of social welfare because the time period matters. In other words, if a portion of the households are working and another portion is retired, they have different utility functions. For this reason, there is no guarantee that the Diamond model's equilibrium is Pareto efficient. Romer shows this in the log utility/Cobb-Douglas production case. Ideally, the social planner can reallocate consumption to make sure that the highest social welfare (utility shared by all households combined) is as high as possible. Unfortunately, it is not possible to shift consumption from the retired to the working (and vice versa) because these consumption bundles are treated as different goods for modeling purposes. The timing of the model creates the possibility of efficiency, known as dynamic inefficiency.

The Diamond model shows that an economy's equilibrium may be dynamically inefficient. So, do economies from dynamic inefficiency? One way to test this is to compare the return on capital (measured by the real interest rate on short-term government debt) and the growth rate of the economy. The golden rule capital stock occurs when the marginal product of capital per effective worker (adjusted for depreciation) is equal to the sum of the growth rates of population and technological progress:

$$k_{GR} \Longrightarrow f'(k^*) - \delta = n + g$$

One difficulty in this simple test is that investment in capital is not risk-free (as assumed in the model). Adjusting for risk and depreciation, it appears that the G-7 countries are dynamically efficient.

2.5 Government in the Diamond Model

The Diamond model is a natural model to use for looking at the implications of government tax/savings policy. Allowing for two different types of households allows us to understand the differential effects of such policies. For simplicity, Romer focuses on the log utility case where the savings rate is constant.

Suppose the government introduces a program that makes savings compulsory for households. The government collects a lump-sum tax G from households when they are working in period (1) and returns the funds to these households (plus interest) $G(1+r_{t+1})$ when they are retired in period (2). Consider how this affects the lifetime budget constraint. Starting with the definition of second-period consumption, we observe that a portion G is deducted from savings because of the lump-sum tax, but $(1+r_{t+1})G$ is available for the retired household to consume:

$$\begin{aligned}
C_{2t+1} &= (1+r_{t+1})(A_t w_t - C_{1t} - G) + (1+r_{t+1})G \\
C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} &= A_t w_t
\end{aligned}$$

Notice that this collapses to the same lifetime budget constraint in the model above. This makes sense because the households are going to save the same fraction of their before- and after-tax income. The level of private savings will be lower because the government is taking a portion of the household's savings as part of the mandatory program. If the government does not invest these funds into the economy's capital stock, then the steady state capital stock per effective worker will be lower. Permanent changes in the lump-sum tax G will affect the steady state, but do not affect the growth rate of per capita income. Temporary changes in G will not affect outcomes because households know the value will return to its initial level and will consume and save based on the long-run value of G.

It is important to note that this is not the Social Security System in the United States. Instead of each generation financing its own retirement, the working generation pays for it with taxes. This suggests that the amount collected by the retired households depends not only on the interest rate, but on the population growth rate.