**ECON 200A**
Spring 2008

**Growth Rates in the Solow Growth Model**

# Model

Assume the following model for the economy:

$$\begin{aligned} Y &= F(K, AL) \\ \frac{\dot{L}}{L} &= n \qquad \frac{\dot{A}}{A} = g \end{aligned}$$

The growth rate in the labor force $n$ and growth rate in technology $g$ are taken as given. In per-effective unit of labor terms, the production function is:

$$\widetilde{y} = f(\widetilde{k})$$

Capital evolves according to:

$$\dot{\widetilde{k}} = sf(\widetilde{k}) - (n + g + \delta)\widetilde{k}$$

where $\dot{k}$ is the derivative of capital with respect to time.[1]
At steady state, $\dot{k} = 0$ :

$$sf(\widetilde{k}) = (n + g + \delta)\widetilde{k}$$

# Growth Rates

## Growth rate of capital $\frac{\dot{K}}{K}$

Capital per-effective unit of labor

$$\dot{\widetilde{k}} = \frac{K}{AL}$$

Taking the derivative with respect to time:

$$\begin{aligned} \dot{\widetilde{k}} &= \frac{\dot{K}(AL) - K(A\dot{L} + L\dot{A})}{(AL)^2} \\ \dot{\widetilde{k}} &= \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A} \\ \dot{\widetilde{k}} &= \frac{\dot{K}}{AL} - \frac{K}{AL}(n + g) \end{aligned}$$

At steady state $\dot{\widetilde{k}} = 0$ :

$$\begin{aligned} \frac{\dot{K}}{AL} &= \frac{K}{AL}(n + g) \\ \frac{\dot{K}}{K} &= (n + g) \end{aligned}$$

So the growth rate of capital is equal to the population growth rate plus the growth rate of technical progress.

[1] $\dot{k}$ is a continuous-time version of $\Delta k$. The change in capital $\Delta k$ is measuring how much capital changes from one period to the next. Rather than assuming discrete periods of time (period 1, 2, etc.), the Solow model uses continuous time. This is why the impulse responses we draw are smooth functions (rather than ones that jump with each discrete unit of time).

**Growth rate of capital per worker** $\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right)$

From the expression above:

$$\dot{\tilde{k}} = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

At steady state $\dot{\tilde{k}} = 0$ :

$$\dot{K} = K\frac{\dot{L}}{L} + K\frac{\dot{A}}{A}$$

Divide by $K$ :

$$\begin{aligned} \frac{\dot{K}}{K} &= \frac{\dot{L}}{L} + \frac{\dot{A}}{A} \\ \frac{\dot{K}}{K} - \frac{\dot{L}}{L} &= \frac{\dot{A}}{A} = g \end{aligned}$$

Therefore, capital per-worker grows at a rate equal to the rate of technical progress.

You can find the remaining growth rates by take the time derivatives for output, consumption, wages, and the real rental rate. You can use the steady state condition (since capital per effective-labor unit is not growing, it must be that $\dot{\tilde{y}} = 0$ and$\dot{\tilde{c}} = 0$.

**Growth rate of output per worker** $\left(\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}\right)$

We know $\frac{\dot{L}}{L} = n$, this is given. We also know that $\dot{y} = 0$ when the economy is at steady state. We can take the time derivative of the per-effective worker output and use this information to solve for $\frac{\dot{Y}}{Y}$. Take the time derivative of $y = \frac{Y}{AL}$

Take the time derivative:

$$\begin{aligned} \dot{\tilde{y}} &= \dot{Y}\frac{1}{AL} + \left(\frac{-Y}{A^2L}\dot{A} + \frac{-Y}{AL^2}\dot{L}\right) \\ &= \dot{Y}\frac{1}{AL} + \frac{-Y}{AL}\frac{\dot{A}}{A} + \frac{-Y}{AL}\frac{\dot{L}}{L} \end{aligned}$$

Use the steady state condition that implies $\dot{y} = 0$ :

$$\begin{aligned} \dot{Y}\frac{1}{AL} &= \frac{Y}{AL}\frac{\dot{A}}{A} + \frac{Y}{AL}\frac{\dot{L}}{L} \\ \frac{\dot{Y}}{Y} &= \frac{\dot{A}}{A} + \frac{\dot{L}}{L} \\ \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} &= \frac{\dot{A}}{A} = g \end{aligned}$$

Therefore, output per-worker grows at a rate equal to the rate of technical progress

## Growth rate of factor payments

### Real wage

Wages are defined as the marginal product of labor. The marginal product of labor is $w \equiv \frac{\partial Y}{\partial L}$. Note, the steady state is defined in per effective units of labor, so the production function is $y = f(k)$:

$$\begin{aligned} \frac{Y}{AL} &= f\left(\frac{K}{AL}\right) \\ Y &= ALf(\widetilde{k}) \end{aligned}$$

Taking the derivative with respect to labor:

$$\begin{aligned} \frac{\partial Y}{\partial L} &= Af(\widetilde{k}) - ALf'\left(\widetilde{k}\right)\frac{K}{AL^2} \\ &= Af(\widetilde{k}) - Af'\left(\widetilde{k}\right)\frac{K}{AL} \\ &= Af(\widetilde{k}) - Af'(\widetilde{k})k \\ &= A\left[f(\widetilde{k}) - f'(\widetilde{k})k\right] \end{aligned}$$

Now, differentiate with respect to time:

$$\dot{w} = \dot{A}\left[f(\widetilde{k}) - f'(\widetilde{k})\widetilde{k}\right] + A[f'(\widetilde{k})\dot{\widetilde{k}} - f''(\widetilde{k})\dot{\widetilde{k}}\widetilde{k} - f'(\widetilde{k})\dot{\widetilde{k}}]$$

The growth rate of the real wage $\frac{\dot{w}}{w}$ is therefore:

$$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{f'(\widetilde{k})\dot{\widetilde{k}} - f''(\widetilde{k})\dot{\widetilde{k}}\widetilde{k} - f'(\widetilde{k})\dot{\widetilde{k}}}{f(\widetilde{k}) - f'(\widetilde{k})\widetilde{k}} = \frac{\dot{A}}{A} + \frac{-f''(\widetilde{k})\dot{\widetilde{k}}\widetilde{k}}{f(\widetilde{k}) - f'(\widetilde{k})\widetilde{k}}$$

Since $\dot{\widetilde{k}} = 0$ at steady state:

$$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} = g$$

### Real rental rate

The rental rate of capital is equal to $r \equiv \frac{\partial Y}{\partial K} - \delta$ (the factor payment is made net of depreciation). Note, $Y = ALf(\widetilde{k}) = AL \times F\left(\frac{K}{AL}\right)$ :

$$\frac{\partial Y}{\partial K} = ALf'(\widetilde{k})\frac{1}{AL} = f'(\widetilde{k})$$

Therefore, the rental rate (net of depreciation) is:

$$r = f'(\widetilde{k}) - \delta$$

Taking the time derivative (the depreciation rate is constant) and dividing through by $r = f'(\widetilde{k}) - \delta$ :

$$\frac{\dot{r}}{r} = \frac{f''(\widetilde{k})\dot{\widetilde{k}}}{f'(\widetilde{k}) - \delta}$$

At steady state, $\dot{\widetilde{k}} = 0$ :

$$\frac{\dot{r}}{r} = 0$$