0.1 Continuous Functions on Intervals

Definition 0.1.1. A function \( f : A \to \mathbb{R} \) is said to be bounded on \( A \) if there exists a constant \( M > 0 \) such that \( |f(x)| \leq M \) for all \( x \in A \).

Remark 0.1.2. A function is bounded if the range of the function is a bounded set of \( \mathbb{R} \). A continuous function is not necessarily bounded. For example, \( f(x) = \frac{1}{x} \) with \( A = (0, \infty) \). But it is bounded on \([1, \infty)\).

Theorem 0.1.3. Let \( I = [a, b] \) be a closed bounded interval, and \( f : I \to \mathbb{R} \) be continuous on \( I \). Then \( f \) is bounded on \( I \).

Proof. Suppose that \( f \) is not bounded on \( I \). Then for each \( n \in \mathbb{N} \), there exists \( x_n \in I \) such that \( |f(x_n)| > n \). As \( x_n \in I \), so \( \{x_n\} \) is bounded, by Bolzano-Weierstrass Theorem, there exists an accumulation point of \( \{x_n\} \), so there exists a subsequence of \( \{x_{n_k}\} \) so that \( x_{n_k} \to x \). Since \( a \leq x_{n_k} \leq b \), we also have \( a \leq x \leq b \), i.e., \( x \in I \). Since \( f \) is continuous on \( I \), we must have \( f(x_{n_k}) \to f(x) \). But this is a contradiction since \( |f(x_{n_k})| > n_k \geq k, k \in \mathbb{N} \). \( \square \)

Remark 0.1.4. In the proof, we use the result of earlier homework: if \( x_n \leq b \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} x_n \leq b \). Similarly, if \( x_n \geq a \) for all \( n \), then \( \lim_{n \to \infty} x_n \geq a \).

Suppose that \( f : S \to \mathbb{R} \). Then define the supremum of \( f \) on \( S \), denoted \( \sup_S f \), to be
\[
\sup_S f = \sup\{f(x) : x \in S\},
\]
similarly, the infimum of \( f \) on \( S \) is defined by
\[
\inf_S f = \inf\{f(x) : x \in S\}.
\]

Note that \( \sup_S f \) could be \( +\infty \) and \( \inf_S f \) could be \( -\infty \), depending on the function \( f \) and \( S \). For example, \( f(x) = x^2 \) and \( S = \mathbb{R} \). Then \( \sup_S f = +\infty \) and \( \inf_S f = 0 \). Now if \( S = (0, 2) \), then \( \sup_S f = 4 \) and \( \inf_S f = 0 \). There are no points in \((0, 2)\) where \( f \) takes 0 and 4.

Remark 0.1.5. If there is \( c \in S \) such that \( \sup_S f = f(c) \), then \( f \) has an absolute maximum on \( S \) at \( c \). Similarly for absolute minimum.
**Theorem 0.1.6.** *(Maxima-Minimum Theorem)* Let \([a, b]\) be a closed and bounded interval, and \(f : I \rightarrow \mathbb{R}\) be continuous on \(I\). Then \(f\) has an absolute maximum and minimum, i.e., there exist points \(c, d \in I\) such that

\[
f(c) = \sup_I f(x) \quad \text{and} \quad f(d) = \inf_I f.
\]

**Proof.** As \(f\) is continuous on \(I\), so it is bounded. Hence both \(\inf f\) and \(\sup f\) exist. Thus, for each \(n \in \mathbb{N}\), there is \(x_n \in I\) such that

\[
\sup_I f - \frac{1}{n} < f(x_n) \leq \sup_I f.
\]

So we have a sequence \(\{x_n\} \subset I\), by Bolzana-Weierstrass, there exists a subsequence \(x_{n_k} \to c\), so \(f(x_{n_k}) \to f(c)\). But the limit is unique. Hence \(f(c) = \sup_I f\). Similarly we can prove that \(f(d) = \inf_I f\). \(\square\)

**Theorem 0.1.7.** *(Bolzano’s Intermediate Value Theorem)* Let \(f\) be a continuous function on \([a, b]\) such that \(f(a) \neq f(b)\). Let \(y\) be any real number between \(f(a)\) and \(f(b)\). Then there is an \(c \in (a, b)\) such that \(f(c) = y\).

**Proof.** Without loss of generality, consider \(f(a) < y < f(b)\). First define a set

\[
S = \{x \in [a, b] : f(x) < y\}.
\]

Thus, \(S \neq \emptyset\) as \(a \in S\). It is clear that \(S\) is bounded. So \(\sup S\) exists, let \(c = \sup S\). Now we prove that \(f(c) = y\). It is clear that \(a \leq c \leq b\).

Suppose that \(c = a\). As \(a \in S\) and \(f\) is continuous at \(a\), so \(f(a) < y\) which implies there exists a \(\delta > 0\) such that \(\forall x \in [a, a + \delta) \implies f(x) < y\). Each point in this neighborhood is in \(S\). This is a contradiction to \(c = \sup S\).

As \(c = \sup S\), there exists a sequence \(\{x_n\}\) in \(S\) such that \(x_n \to c\). From \(f(x_n) < y \implies f(c) \leq y\), as \(f\) is continuous at \(c\).

If \(f(c) < y\), again by \(f\) being continuous at \(c\), there is a neighborhood of \(c\), \((c - \delta, c + \delta)\), such that \(f(x) < y\) for all \(x \in (c - \delta, c + \delta)\), which is contradiction, as \(c = \sup S\). \(\square\)

**Example 0.1.8.** Consider \(f(x) = x^2 - 2\) on \([0, 2]\). So \(f(0) = -2, f(2) = 2\). Let \(y = 0\). Then from the theorem, there exists \(c \in (0, 2)\) such that \(f(c) = 0\), in fact, \(c = \sqrt{2}\).

One note about this: if we only consider the set of rationals, then the graph of \(x^2 - 2\) would cross the \(x\)-axis without meeting it. Another example of the set of real numbers complete (axiom 12).
Corollary 0.1.9. Let $f$ be a continuous function on $[a,b]$ and define $m = \inf f$ and $M = \sup f$. Then the range of $f$ is the interval $[m,M]$, i.e., $f([a,b]) = [m,M]$.

Proof. We know from above, there exists $c,d \in [a,b]$ such that $f(c) = m, f(d) = M$. And any number $y \in (m,M)$, there is $c \in (a,b)$ such that $f(c) = y$. From the definition $m,M$, $f$ does not have values outside $[m,M]$. So the range equals $[m,M]$.

Theorem 0.1.10. Let $I$ be an interval and let $f : I \to \mathbb{R}$ be continuous on $I$. If $\alpha < \beta$ are numbers in $I$ such that $f(\alpha) < 0 < f(\beta)$ (or $f(\alpha) > 0 > f(\beta)$), then there exists a number $c \in (\alpha, \beta)$ such that $f(c) = 0$.

Proof. As $f$ is continuous on $I$, so $f$ is continuous on $(\alpha, \beta)$. Apply the Intermediate Value Theorem with $y = 0$.

Example 0.1.11. Let $f(x) = \frac{1}{x^2+1}$.

1. $I_1 = (-1, 1)$. $f(I_1) = (\frac{1}{2}, 1]$.

2. $I_2 = [0, \infty)$, $f(I_2) = (0, 1]$.

Lemma 0.1.12. Let $S \subseteq \mathbb{R}$ be a nonempty set with the property if $x, y \in S$ with $x < y$, then $[x, y] \subseteq S$. Then $S$ is an interval.

Theorem 0.1.13. Let $I$ be an interval and let $f : I \to \mathbb{R}$ be continuous on $I$. Then $f(I)$ is an interval.

0.2 Uniform Continuity

First recall the definition of $f$ being continuous at $x_0$: $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x : |x - x_0| < \epsilon \implies |f(x) - f(x_0)| < \epsilon$.

In general, $\delta$ depends on both $\epsilon$ and $x_0$, as function changes rapidly at some points and flat at some other points. We start some examples to look into this.

Example 0.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 2x$. Let $x_0 \in \mathbb{R}$. Consider

$$|f(x) - f(x_0)| = |2x - 2x_0| = 2|x - x_0|.$$

From this we can see if we choose $\delta = \epsilon/2$, we have $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. In this case, $\delta$ depends only on $\epsilon$, it works for all $x_0 \in \mathbb{R}$. 

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Example 0.2.2. Let \( f : (0, \infty) \to \mathbb{R} \) with \( f(x) = 1/x \). Let \( x_0 = u > 0 \).
Consider
\[
|f(x) - f(u)| = \frac{|x - u|}{xu}.
\]
As \( x \to u \), so consider only \( |x - u| < u/2 \), i.e., \( u/2 < x < 3u/2 \). Then \( 1/x < 2/u \). Hence \( 1/xu < (1/u)(2/u) = 2/u^2 \). Now given \( \epsilon > 0 \), choose \( \delta = \min\{u/2, u^2\epsilon/2\} \). So when \( |x - u| < \delta \), \( |f(x) - f(u)| < \epsilon \).

Here \( \delta \) depends on both \( \epsilon \) and \( u \). In fact, there is no \( \delta \) for all \( u > 0 \), as then \( \delta = 0 \).

See the graph of \( f(x) = 1/x \).

Definition 0.2.3. Let \( f : D \to \mathbb{R} \) is uniformly continuous on \( E \subset D \) iff \( \forall \epsilon > 0 \exists \delta > 0 \ \forall x, y \in E, |x - y|, \delta \Rightarrow |f(x) - f(y)| < \epsilon \). If \( f \) is uniformly continuous on \( D \), then \( f \) is uniformly continuous.

Remark 0.2.4. \( f \) uniformly continuous on \( E \) implies \( f \) is continuous on \( E \). The converse is not true.

Example 0.2.5.  
1. \( f : [2.5, 3] \to \mathbb{R} \) defined by \( f(x) = \frac{3}{x-2} \).
2. \( f : (0, 6) \to \mathbb{R} \) with \( f(x) = x^2 + 2x - 5 \).
3. \( f : (2, 3) \to \mathbb{R} \) with \( f(x) = \frac{3}{x-2} \).

Non-uniform Continuity Criteria Let \( A \subset \mathbb{R} \) and let \( f : A \to \mathbb{R} \). Then the following statements are equivalent.

1. \( f \) is not uniformly continuous on \( A \).
2. \( \exists \epsilon_0 > 0 \) such that for every \( \delta > 0 \) there are points \( x_\delta, y_\delta \in A \) such that \( |x_\delta - y_\delta| < \delta \) and \( |f(x_\delta) - f(y_\delta)| \geq \epsilon_0 \).
3. \( \exists \epsilon_0 > 0 \) and two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( A \) such that \( \lim(x_n - y_n) = 0 \) and \( |f(x_n) - f(y_n)| \geq \epsilon_0 \) for all \( n \in \mathbb{N} \).

Example 0.2.6. Let \( f : (0, \infty) \to \mathbb{R} \) with \( f(x) = 1/x \). Now pick \( \epsilon_0 = 1/2 \), and choose \( x_n = 1/n \) and \( y_n = 1/(n+1) \). Then \( \lim(x_n - y_n) = 0 \) and \( |f(x_n) - f(y_n)| = 1 > 1/2 \) for all \( n \).

Theorem 0.2.7. Let \( I \) be a closed bounded interval and let \( f : I \to \mathbb{R} \) be continuous on \( I \). Then \( f \) is uniformly continuous on \( I \).
Proof. If $f$ is not uniformly continuous on $I$. From the above, $\exists \epsilon_0 > 0$ and $x_n, y_n \in I$ such that $x_n - y_n \to 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n$. As $I$ is bounded, so by Bolzana-Weierstrass, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to $z \in I$, as $I$ is closed interval. In addition, from

$$|y_{n_k} - z| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$y_{n_k} \to z$ as well.

Now as $f$ is continuous at $z$, so we have $f(x_{n_k}) \to f(z)$ and $f(y_{n_k}) \to f(z)$. But this is a contradiction, as $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n$. \[\square\]

Lipschitz Functions

**Definition 0.2.8.** Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \forall x, y \in A,$$

then $f$ is said to be a Lipschitz function (or to satisfy a Lipschitz condition) on $A$.

Geometrically, $f$ is Lipschitz if and only if the slopes of secant line joining points $(x, f(x))$ and $(y, f(y))$ are bounded by $K$.

**Theorem 0.2.9.** Let $f : A \to \mathbb{R}$ be a Lipschitz function, then $f$ is uniformly continuous on $A$.

**Proof.** Let $\epsilon > 0$, choose $\delta = \epsilon/K$. Then for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. \[\square\]

**Example 0.2.10.** Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. $f$ is uniformly continuous on $[a, b]$ but not on $\mathbb{R}$.

**Proof.** Let $c \in \mathbb{R}$. Consider

$$|f(x) - f(c)| = |x^2 - c^2| = |x - c||x + c|.$$

As $x$ is close to $c$, we assume that $|x - c| < 1$. So this implies $|x| < 1 + |c|$, thus $|x + c| \leq 1 + 2|c|$. Hence

$$|x - c||x + c| < |x - c|(1 + 2|c|).$$
Now for $\epsilon > 0$, choose $\delta = \min\{1, \frac{\epsilon}{1+2|c|}\}$ such that for all $x$ satisfying $|x-c| < \delta$, $|f(x) - f(c)| < \epsilon$, i.e., $f$ is continuous on $\mathbb{R}$.

As $\delta$ depends on both $\epsilon$ and $c$, $c$ is larger and larger, the values of $\delta$ is smaller and smaller (as the graph becomes more steeper). There is no such $\delta$ that works for all points. In fact, $\inf_c \delta = 0$.

But when we consider only on $[-a, a]$ for $a > 0$. Then

$$|x + c| \leq |x| + |c| \leq 2a,$$

hence $\delta = \epsilon/2a$ works for all points on $[-a, a]$, i.e., $f$ is uniformly continuous on $[-a, a]$. 

\[ \square \]