4. Let \( R \) be the relation “congruence modulo 5” defined on \( \mathbb{Z} \) as follows: \( x \) is congruent to \( y \) modulo 5 if and only if \( x - y \) is a multiple of 5, and we write \( x \equiv y \pmod{5} \).

(a) Prove that “congruence modulo 5” is an equivalence relation.

**Proof.** Let \( x \in \mathbb{Z} \). Then \( x - x = 0 = 5(0) \). So \( 5|x - x \), hence \( x \equiv x \pmod{5} \).

Let \( x, y \in \mathbb{Z} \) such that \( x \equiv y \pmod{5} \). Hence \( 5|x - y \) and \( 5|y - x \). Therefore \( 5|x - y + (y - x) = 5l = 5(0) \). So \( 5|x - y \), hence \( x \equiv y \pmod{5} \).

Therefore equivalence modulo 5 is an equivalence relation.

6. Prove that \( R \) is an equivalence relation, where \( R \) is a relation on \( \mathbb{Z} \) given by the following.

\( xRy \) if and only if \( x^2 + y^2 \) is a multiple of 2.

**Proof.** Let \( x \in \mathbb{Z} \). Then \( x^2 + x^2 = 2x^2 \). So \( x^2 + x^2 \) is a multiple of 2, and hence \( xRx \). Therefore \( R \) is reflexive.

Suppose \( xRy \) for some \( x, y \in \mathbb{Z} \). Then \( x^2 + y^2 = 2k \) for some \( k \in \mathbb{Z} \). Therefore \( y^2 + x^2 = 2k \) and hence \( yRx \). Therefore \( R \) is symmetric.

Suppose \( xRy \) and \( yRz \) for some \( x, y, z \in \mathbb{Z} \). Then \( x^2 + y^2 = 2k \) and \( y^2 + z^2 = 2l \) for some \( k, l \in \mathbb{Z} \). Therefore \( x^2 + z^2 = (2k - y^2) + (2l - y^2) = 2(k + l - y^2) \).

Thus \( x^2 + z^2 \) is a multiple of 2 and hence \( xRz \). Therefore \( R \) is transitive.

Thus \( R \) is an equivalence relation.

12(a) Let \( A = \mathbb{R} \setminus \{0\} \) and consider the relation \( R \) on \( A \times A \) given by \( (a, b)R(c, d) \) iff \( ad = bc \). Determine whether or not \( R \) is an equivalence relation.

Claim: \( R \) is an equivalence relation.

**Proof.** Let \( (a, b) \in A \times A \), then \( ab = ab \), so \( (a, b)R(a, b) \). Hence \( R \) is reflexive.

Suppose \( (a, b)R(c, d) \). Then \( ad = bc \) and so \( cb = da \). Thus \( (c, d)R(a, b) \), so \( R \) is symmetric.

Suppose \( (a, b)R(c, d) \) and \( (c, d)R(e, f) \). Then \( ad = bc \) and \( cf = de \). Thus \( af = \frac{bc}{c} \cdot \frac{de}{a} = be \) and hence \( (a, b)R(e, f) \). Thus \( R \) is transitive.

Therefore \( R \) is an equivalence relation.

16. Let \( \mathcal{P}(A) \) be the power set of the nonempty set \( A \), and let \( C \) denote a fixed subset of \( A \). Define \( R \) on \( \mathcal{P}(A) \) by \( xRy \) iff \( x \cap C = y \cap C \). Prove that \( R \) is an equivalence relation.

**Proof.** Let \( x \in \mathcal{P}(A) \). Then \( x \cap C = x \cap C \), so \( xRx \). Thus \( R \) is reflexive.

Suppose \( xRy \). So \( x \cap C = y \cap C \), thus \( y \cap C = x \cap C \). So \( yRx \) and hence \( R \) is symmetric.

Suppose \( xRy \) and \( yRz \). So \( x \cap C = y \cap C \) and \( y \cap C = z \cap C \). Thus \( x \cap C = z \cap C \), so \( xRz \). Thus \( R \) is transitive.

Therefore \( R \) is an equivalence relation.

22(a) Prove \( R \) is symmetric if and only if \( R = R^{-1} \).

**Proof.**
(⇒) Assume $R$ is symmetric.

\[(a, b) \in R \Leftrightarrow aRb \Leftrightarrow bRa \Leftrightarrow aR^{-1}b \Leftrightarrow (a, b) \in R^{-1}\]

Thus $R = R^{-1}$.

(⇐) Assume $R = R^{-1}$.

\[aRb \Rightarrow (a, b) \in R \Rightarrow (a, b) \in R^{-1} \Rightarrow (b, a) \in R^{-1} \Rightarrow (b, a) \in R\]

Hence $R$ is symmetric.

\[
28. \text{ Suppose that } f \text{ is an onto mapping from } A \text{ to } B. \text{ Prove that if } \{B_\lambda\}, \lambda \in \mathcal{L} \text{ is a partition of } B, \text{ then } \{f^{-1}(B_\lambda)\}, \lambda \in \mathcal{L} \text{ is a partition of } A.
\]

\textbf{Proof.} Let $f : A \to B$ be an onto mapping and let $\{B_\lambda | \lambda \in \mathcal{L}\}$ be a partition of $B$.

First we need to show that $f^{-1}(B_\lambda) \neq \emptyset$ for all $\lambda \in \mathcal{L}$. Since $\{B_\lambda | \lambda \in \mathcal{L}\}$ is a partition, we know $B_\lambda \neq \emptyset$ for all $\lambda \in \mathcal{L}$. Let $b \in B_\lambda$. Since $f$ is onto there exists $a \in A$ such that $f(a) = b$. So $f(a) \in B_\lambda$ and thus $a \in f^{-1}(B_\lambda)$. Therefore $f^{-1}(B_\lambda) \neq \emptyset$ for all $\lambda \in \mathcal{L}$.

Let $a \in A$. Then there exists $b \in B$ such that $f(a) = b$. Moreover $b \in B_\gamma$ for some $\gamma \in \mathcal{L}$, since $\{B_\lambda\}$ is a partition. Therefore $f(a) \in B_\gamma$, so $a \in f^{-1}(B_\gamma)$. Therefore $\bigcup_{\lambda \in \mathcal{L}} f^{-1}(B_\lambda) = A$.

Now, suppose $f^{-1}(B_\alpha) \cap f^{-1}(B_\beta) \neq \emptyset$. Thus there exists $a \in f^{-1}(B_\alpha) \cap f^{-1}(B_\beta)$. So $a \in f^{-1}(B_\alpha)$ and $a \in f^{-1}(B_\beta)$. Hence $f(a) \in B_\alpha$ and $f(a) \in B_\beta$. Thus $f(a) \in B_\alpha \cap B_\beta$. Therefore $B_\alpha \cap B_\beta \neq \emptyset$ and thus $B_\alpha = B_\beta$ since $\{B_\lambda | \lambda \in \mathcal{L}\}$ is a partition. Hence $f^{-1}(B_\alpha) = f^{-1}(B_\beta)$.

Therefore $\{f^{-1}(B_\lambda) | \lambda \in \mathcal{L}\}$ forms a partition of $A$. 
\[
\blacksquare
\]