Homework #0 – Math 210B
Not due

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1. Recall that if G is a group, then $Aut(G)$ is the set of all automorphisms on G. In addition, $Aut(G)$ is a group with respect to composition.
   (a) Let $V = \{e, a, b, ab\}$ be the Klein-4 group. Prove that $Aut(V) = S_3$.
   (b) Prove that $Aut(S_3) = S_3$.
   (c) Prove that $Aut(\mathbb{Z}) = \mathbb{Z}_2$.
   (d) Prove that $|Aut(G)| = 1$, then $|G| \leq 2$.

2. Let R be a commutative ring with unity. Define $\mathcal{F}(R)$ to be the set of all functions from R to R. Let $f, g \in \mathcal{F}(R)$. Define addition on this set by $(f+g)(r) = f(r)+g(r)$ and multiplication on this set by $(fg)(r) = f(r)g(r)$.
   (a) Prove that $\mathcal{F}(R)$ is a commutative ring with unity.
   (b) Prove that $\mathcal{F}(R)$ is not an integral domain.
   (c) Prove that $f + f = 0$ for all $f \in \mathcal{F}(\mathbb{Z}_2)$.

3. Let R be a ring. We define $R[a]$ to be the smallest ring containing R and a.
   (a) Prove $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.
   (b) Prove that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.
   (c) Prove that $\mathbb{Z}[\sqrt[3]{2}] \neq \{a + b\sqrt[3]{2} \mid a, b \in \mathbb{Z}\}$.

4. Let R be a commutative ring with unity.
   (a) A non-zero element $a \in R$ is called nilpotent if $a^n = 0$ for some $n \in \mathbb{Z}^+$. Prove that if a is nilpotent in R, then $1 + a$ is a unit in R.
   (b) Prove the set of all nilpotent elements in R is an ideal in R.

5. Let R be a commutative ring with unity.
   (a) If I is an ideal in R, we define the radical of I in the following way.
   $$\sqrt{I} = \{r \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}^+\}$$
   Prove $\sqrt{I}$ is an ideal in R.
   (b) Let $a \in R$, then we define the annihilator of a in the following way.
   $$Ann(a) = \{r \in R \mid ra = 0\}$$
   Prove that $Ann(a)$ is an ideal in R.