Appendix 3: Coordinate Transformations

Transformation of coordinates from one Cartesian system to another that is translated and rotated from the first one comes up frequently in 3D graphics. This section provides a quick way of computing such transformations. This method is explained in greater details, for example in (4)

In our notation, the superscript indicates the coordinate system in which the measurement is being made and "{}" are used to denote the coordinate systems. In **Error! Reference source not found.** {A} and {B} are two coordinate systems. ${}^{A}\vec{p}$ refers to point *p* as observed from coordinate system {A}.

3.1 Pure Rotation of coordinate systems

Say the origins of {A} and {B} coincide and that {B} is rotated relative to {A}.

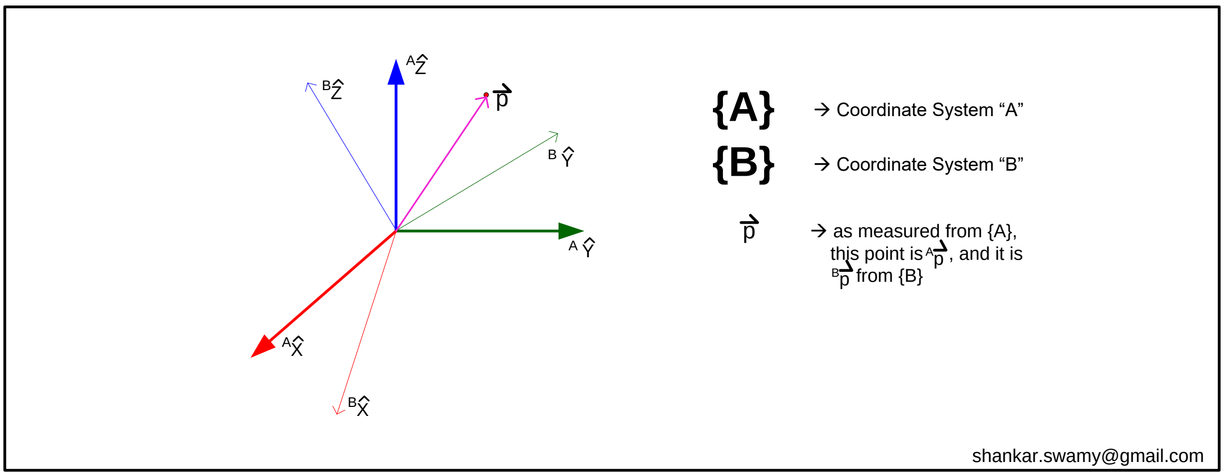

Figure 8-5 Rotation of coordinate systems

We are interested in expressing ${}^{A}\vec{p}$ in terms of ${}^{B}\vec{p}$.

Say ${}^{A}\vec{p} \equiv ({}^{A}p_x, {}^{A}p_y, {}^{A}p_z)$

The components of any vector (or point) in a coordinate system are simply the projections of that vector along the three coordinate axes. So to express the ${}^{A}\vec{p}$ in {B}, we need the projection of the above components along the three axes of {B}, as seen from {A}. The projection of a vector along another one is a dot product between the two vectors. Hence we have:

$${}^{A}\vec{p}_x \equiv ({}^{A}X_B, {}^{A}X_B, {}^{A}X_B).({}^{B}p_x, {}^{B}p_y, {}^{B}p_z)$$

This equation says that the ${}^{A}\vec{p}_x$ component is contributed to by the projections of all three components of ${}^{B}\vec{p}$ along ${}^{A}X_B$.

Similarly we have the other components:

$${}^{A}\vec{p}_y \equiv ({}^{A}Y_B, {}^{A}Y_B, {}^{A}Y_B).({}^{B}p_x, {}^{B}p_y, {}^{B}p_z)$$

$${}^{A}\vec{p}_z \equiv ({}^{A}Z_B, {}^{A}Z_B, {}^{A}Z_B).({}^{B}p_x, {}^{B}p_y, {}^{B}p_z)$$

This set of equations we write as:

$$\begin{bmatrix} {}^{A}\vec{p}_x \\ {}^{A}\vec{p}_y \\ {}^{A}\vec{p}_z \end{bmatrix} = \begin{bmatrix} {}^{A}X_B & {}^{A}X_B & {}^{A}X_B \\ {}^{A}Y_B & {}^{A}Y_B & {}^{A}Y_B \\ {}^{A}Z_B & {}^{A}Z_B & {}^{A}Z_B \end{bmatrix} \begin{bmatrix} {}^{B}\vec{p}_x \\ {}^{B}\vec{p}_y \\ {}^{B}\vec{p}_z \end{bmatrix}$$

We abbreviate as: ${}^{A}\vec{p} = {}^{A}_{B}\mathbf{R} \; {}^{B}\vec{p}$, where

$${}^{A}_{B}\mathbf{R} \equiv \begin{bmatrix} {}^{A}X_B & {}^{A}X_B & {}^{A}X_B \\ {}^{A}Y_B & {}^{A}Y_B & {}^{A}Y_B \\ {}^{A}Z_B & {}^{A}Z_B & {}^{A}Z_B \end{bmatrix}$$

3.1.1 Rotation Matrices

Notice that, for example, ${}^{A}X_B$ is just the projection of the x-axis of {B} on the x-axis of {A}, and so on for the other elements of ${}^{A}_{B}\mathbf{R}$. These will evaluate to either cosine or sine of the angles beteen the respective axes, if the rotations are strictly along on of the principal axes – the so called principle rotations. In each of those cases, the ${}^{A}_{B}\mathbf{R}$ reduce to the following matrices:

$${}^{A}_{B}\mathbf{R}_x \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$${}^{A}_{B}\mathbf{R}_y \equiv \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$${}^{A}_{B}\mathbf{R}_z \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ sin\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These are the principle rotation matrices when {B} is rotated relative to {A} in the counter clockwise direction. (If the {B} was rotated in the clockwise direction, then in the matrices θ should be replaced by $-\theta$.) Given that, a vector as seen in {A} is related to the same vector as seen in {B} by the relation: ${}^{A}\vec{p} = {}^{A}_{B}\mathbf{R} \; {}^{B}\vec{p}$.

Since the rotation matrices are orthogonal, ${}^{A}_{B}\mathbf{R} = {}^{B}_{A}\mathbf{R}^{\mathrm{T}}$, and ${}^{B}\vec{p} \equiv {}^{A}_{B}\mathbf{R}^{\mathbf{T}} \, {}^{A}\vec{p} = {}^{B}_{A}\mathbf{R} \; {}^{A}\vec{p}$.

Since any rotation can be decomposed into a series of planar rotations, the three principal rotation matrices are sufficient to solve a general rotation problem by applying the equation ${}^{A}\vec{p} = {}^{A}_{B}\mathrm{R} \; {}^{B}\vec{p}$, where ${}^{A}_{B}\mathrm{R}$ would be a composite of multiple principle rotation matrices, where {A} is the frame that is rotated and {B} is the original frame.

3.1.2 Example

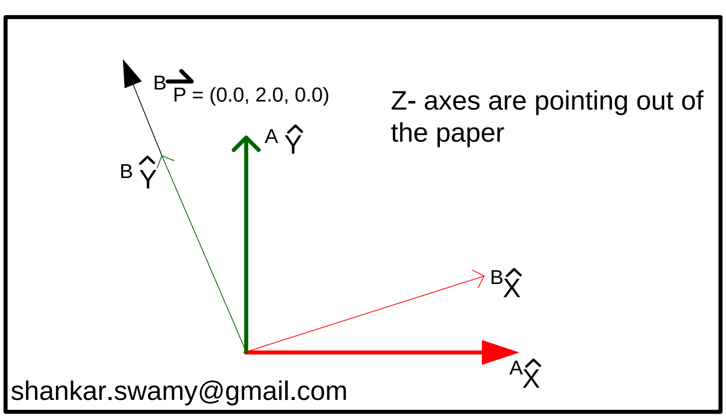

Figure 8-6 Rotated coordinate systems

This example is from (4), p. 23. Consider two coordinate systems {A} and {B} coincident with each other. {B} is rotated by 30 degrees about z-axes in the counter clockwise direction. A vector measured in the rotated frame is given by

${}^{B}\vec{p} = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}$. What is this vector, as measured in the original frame?

We are seeking ${}^{A}\vec{p}$. $\cos 30° = 0.866, \sin 30° = 0.5$. We have,

$$^{A}\vec{p} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ 1.732 \\ 0.0 \end{bmatrix}$$

3.1.3 Useful properties of rotation matrices

Since the principal rotation matrices are orthogonal – their inverses are same as their transposes. So to "undo" a rotation, we mupltiply the resulting vector by the transpose of the rotation matrix.

In Euclidean geometry, rotation is an isometry. That is, it moves points without changing the distances between them. And, as opposed to translation, rotation always leaves one point unchanged.

3.2 Combined rotation and translation of coordinate systems

Figure 4-3 shows coordinate systems {A} and {B} and a point $\vec{p}$. This point measured with reference to {A}, is represented by $^{A}\vec{p}$ and that measured with reference to {B} is represented by $^{B}\vec{p}$. The origin of {B} is at a distance⟦ $^{A}\vec{p}_{BORG}$. Further the principle axes of {B} are also rotated relative to {A}.

{A} and {B} are two different coordinate systems with the origin of {B} displaced from that of {A} by a vector Ap_{BORG}.

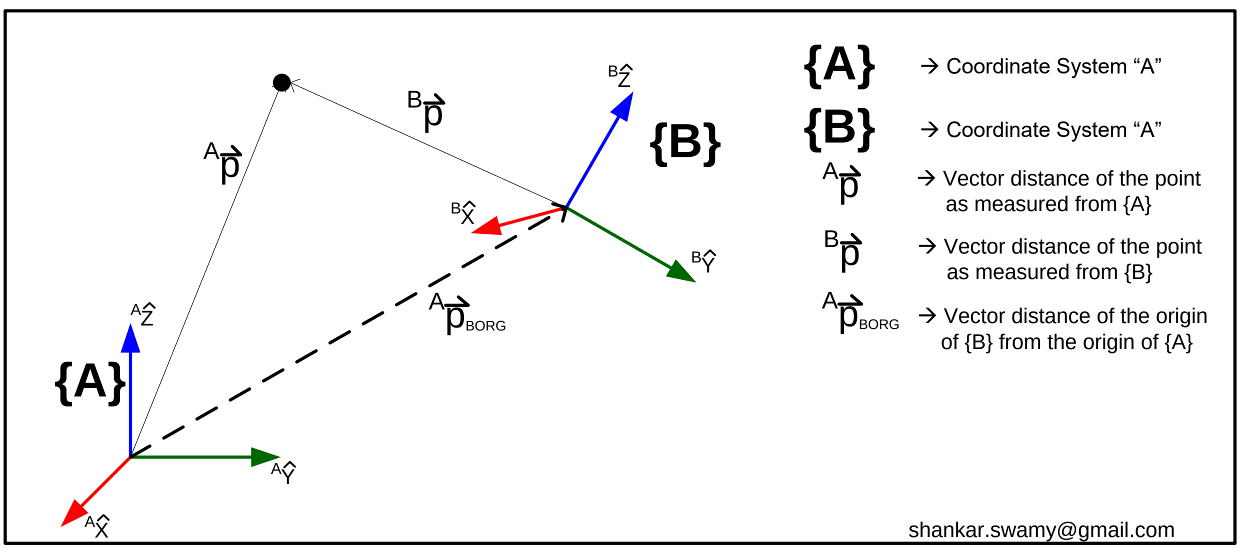

Figure 8-7 Transformation of coordinates

We seek $^{A}\vec{p}$. $^{B}\vec{p}$ is $^{A}_{B}R$ $^{B}\vec{p}$ in {A} and with that, we have all the three vectors connecting the origin of {A}, origin of {B} and the point under question expressed in {A}:

$$^{A}\vec{p} = {}^{A}_{B}R\,{}^{B}\vec{p} + {}^{A}\vec{p}_{PORG}$$

Using matrix-vector equation, we can write the above equation equivalently as:

$$\begin{bmatrix} {}^{A}\vec{p}_x \\ {}^{A}\vec{p}_y \\ {}^{A}\vec{p}_z \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}X_B & {}^{A}X_B & {}^{A}X_B & {}^{A}\vec{p}_{PORG_x} \\ {}^{A}Y_B & {}^{A}Y_B & {}^{A}Y_B & {}^{A}\vec{p}_{PORG_y} \\ {}^{A}Z_B & {}^{A}Z_B & {}^{A}Z_B & {}^{A}\vec{p}_{PORG_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\vec{p}_x \\ {}^{B}\vec{p}_y \\ {}^{B}\vec{p}_z \\ 1 \end{bmatrix}$$

The 4x4 matrix can be cast as the transformation matrix, to rewrite the equation as:

$${}^{A}\vec{p} = {}^{A}_{B}\mathrm{T}\,{}^{B}\vec{p}$$

where, $${}^{A}_{B}\mathrm{T} = \begin{bmatrix} {}^{A}X_B & {}^{A}X_B & {}^{A}X_B & {}^{A}\vec{p}_{PORG_x} \\ {}^{A}Y_B & {}^{A}Y_B & {}^{A}Y_B & {}^{A}\vec{p}_{PORG_y} \\ {}^{A}Z_B & {}^{A}Z_B & {}^{A}Z_B & {}^{A}\vec{p}_{PORG_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} [{}^{A}_{B}\mathrm{R}] & & & {}^{A}\vec{p}_{PORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with,

$$\left[{}^{A}_{B}\mathrm{R}\right] \equiv \begin{bmatrix} {}^{A}X_B & {}^{A}X_B & {}^{A}X_B \\ {}^{A}Y_B & {}^{A}Y_B & {}^{A}Y_B \\ {}^{A}Z_B & {}^{A}Z_B & {}^{A}Z_B \end{bmatrix}$$

and,

$${}^{A}\vec{p}_{PORG} \equiv \begin{bmatrix} {}^{A}\overrightarrow{p_x}_{PORG} \\ {}^{A}\overrightarrow{p_y}_{PORG} \\ {}^{A}\overrightarrow{p_z}_{PORG} \end{bmatrix}.$$

The idea behind writing it this way is to quickly compute ${}^{B}_{A}\mathrm{T}$ from ${}^{A}_{B}\mathrm{T}$, if needed:

$${}^{B}_{A}\mathrm{T} = \begin{bmatrix} [{}^{A}_{B}\mathrm{R}]^T & & & -[{}^{A}_{B}\mathrm{R}]^T\,{}^{A}\vec{p}_{PORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where orthogonal property of the rotational matrices that $[{}^{A}_{B}\mathrm{R}]^{-1} \equiv -[{}^{A}_{B}\mathrm{R}]^T$ has been used.

3.3 Generic Formats of rotation, translation, scaling, perspective projection and shear matrices

3.3.1 Translation

$$\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3.2 Rotation

$$R = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} & 0 \\ R_{yx} & R_{yy} & R_{yz} & 0 \\ R_{zx} & R_{zy} & R_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} -R_{xx} & -R_{yx} & -R_{zx} & 0 \\ -R_{xy} & -R_{yy} & -R_{zy} & 0 \\ -R_{xz} & -R_{yz} & -R_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3.3 Scale

$$\text{S} = \begin{bmatrix} S_{xx} & 0 & 0 & 0 \\ 0 & S_{yy} & 0 & 0 \\ 0 & 0 & S_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} \frac{1}{s_{xx}} & 0 & 0 & 0 \\ 0 & \frac{1}{s_{yy}} & 0 & 0 \\ 0 & 0 & \frac{1}{s_{zz}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.3.4 Perspective Projection

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

with at least one of $r_x,\ r_y\ or\ r_z$ not equal to zero. For example, for perspective along the $\pm z$ –direction, $r_z = r_y \ \equiv 0; r_z \ \neq 0$.