Appendix 2: Linear Transformations & the use of Homongeneous Coordinates

The use of homogeneous coordinates in graphics is partly related to linear transformations and their properties. This section provides a quick review of the two concepts and why homogeneous coordinates preferred.

Matrices represent linear transformations. Linear transformation of vectors conforms to two conditions:

1. $\mathcal{F}(\vec{\alpha} + \vec{\beta}) = \mathcal{F}(\vec{\alpha}) + \mathcal{F}(\vec{\beta})$, where $\vec{\alpha}$ and $\vec{\beta}$ are two arbitrary vectors,
2. $\mathcal{F}(c\,\vec{\alpha}) = c\,\mathcal{F}(\vec{\alpha})$, where *c* is an arbitrary constant.

Any transformation that conforms to above two conditions is a linear transformation and can be represented by a matrix. For a vector of zero length, the second condition leads to $\mathcal{F}(c\,\vec{0}) = c\,\mathcal{F}(\vec{0})$ which valid if and only if$\mathcal{F}\left(c\,\vec{0}\right) \equiv 0$. That is, the origin does not change under linear transformations.

There are three transformations of vectors in the graphics pipeline: rotation, scaling and the translation. From the above conditions that the first two types of transformations are linear in Euclidean space is readily verified. The translation is *not* a linear transformation in the Euclidean space: when you translate a vector, the origin of the vector also moves and that violates the second condition above. So in the Euclidean space, the translation is not a linear transformation and cannot be represented by a matrix in the Euclidean space. However, translation happens to be a linear transformation in the homogeneous space and can be represented by a 4x4 matrix.

This is one of the advantages of using 4x4 matrices in the pipeline. The other advantage is the ease of clipping in the homogeneous space.

Uniformly using the 4x4 matrices throughput the pipeline helps the architects to build a system around a pipeline that is, among other things, is optimized exclusively for operations on such matrices without having to worry about different other matrix orders.

2.1 Types of linear transformations

Based on the properties of the transformation they are classified into different types and a few that are relevant to 3D graphics are listed here.

2.1.1 Projective Transformation

This is the most general of all the transformations mentioned here. Representation of a generic projective transformation in 3D requires homogeneous coordinates and hence a [4x4] matrix. The matrix must be non-singular (that is, the matrix must have a non-zero determinant).

$$\begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix} \qquad [1.1.1]$$

Given that the determinant of the above matrix is non-zero, it represents a projective transformation.

Essentially, any [4x4] matrix that conforms to above conditions represents some projective transformation. Projective transformation can bring points at infinity to finite locations and swing points at finite range to points at infinity. Deductively, they can transform parallel lines or planes into intersecting line or planes and vice-versa.

The so called cross ratio is what is invariant under projective transformation. If a, b, c, and d are four points on a line, then their cross ratio is $\frac{(c-a)/(d-a)}{(c-b)/(d-b)}$

In 3D graphics, with significant exception of perspective projection, we mostly deal with affine transformations which is a proper subset of projective transformation.

2.1.2 Affine Transformation

An *affine* transformation is any transformation that preserves co-linearity and ratios of the distances. Rotations, translations, scaling, shears, their combinations are all affine transformations. A [4x4] matrix that represents an affine transformation necessarily has its fourth column (in our representation) of elements set to 0, 0, 0, and $\alpha \neq 0$, (reading from the top of the column).

$$\begin{bmatrix} a & e & i & 0 \\ b & f & j & 0 \\ c & g & k & 0 \\ d & h & l & \alpha \end{bmatrix} \qquad [1.1.2]$$

Given that $\alpha \neq 0$, and the determinant of the matrix is not zero, this matrix represents an affine transformation.

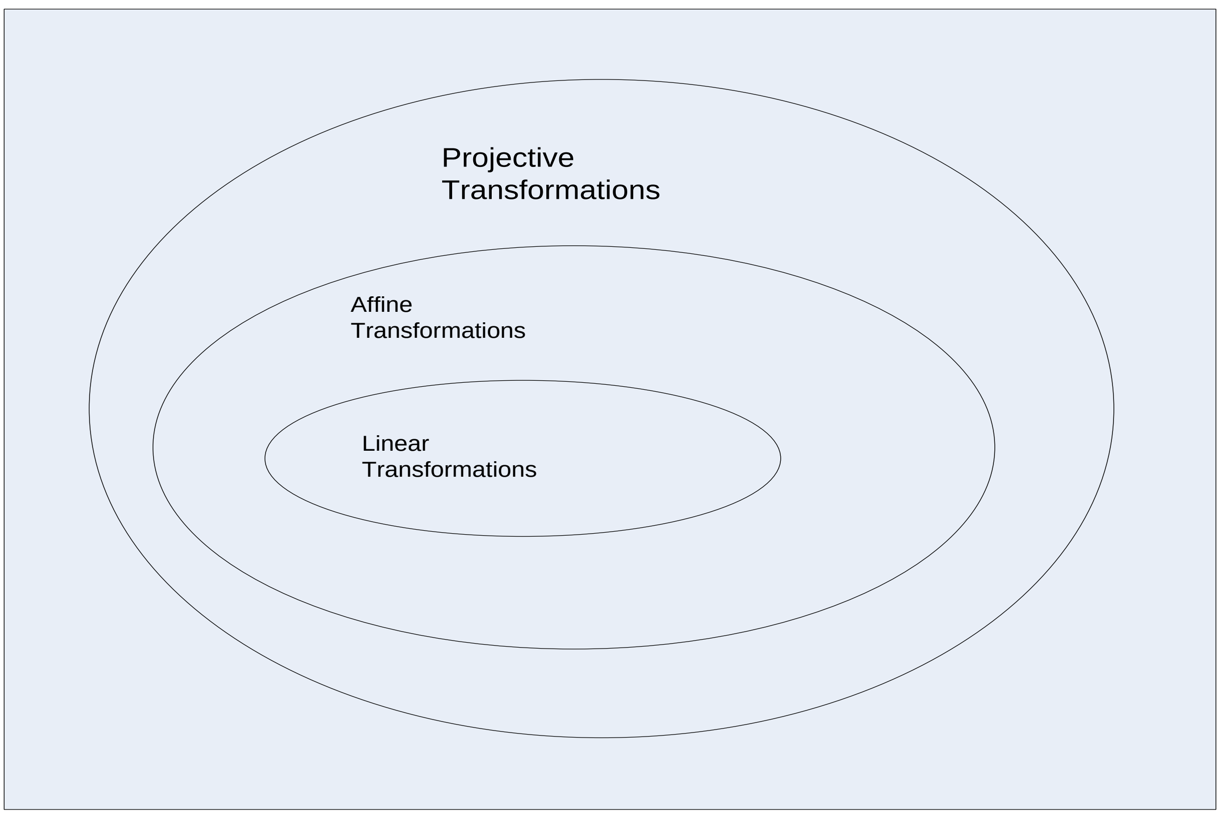

Figure 8-1 Venn diagram showing the relationship between linear, affine and projective transformations

Affine transformations do not alter the degree of the polynomial they transform. So we can expect a sphere to transform into a spheroid or a torus to transform into another distorted torus etc. Parallel lines or planes are retained to be parallel and intersecting lines or planes are retained to be such. But they are not guaranteed to retain the lengths or angle measures. Affine transformations have the general form $f(\vec{u}\,) = M\,\vec{u} + \vec{c}$ where M is an invertible matrix and $\vec{c}$ is a constant vector.

Linear transformations have the form $g(\vec{u}) = M\,\vec{u}$ where M is an invertible matrix. This means that the most generic form of a linear transformation is $(ax + by, cx + dy)$ with four real numbers a, b, c, and d completely specifying the transformation. All linear transformations essentially have to preserve two properties: $F\left(\vec{\alpha} + \vec{\beta}\right) = F(\alpha) + F(\beta)$ and $F(c\,\vec{\alpha}) = c\,F(\alpha)$. Linear transformation include those transformations which do not change the origin – scaling and rotation. Notably they do not include translation.

The affine transformation includes translation as well.

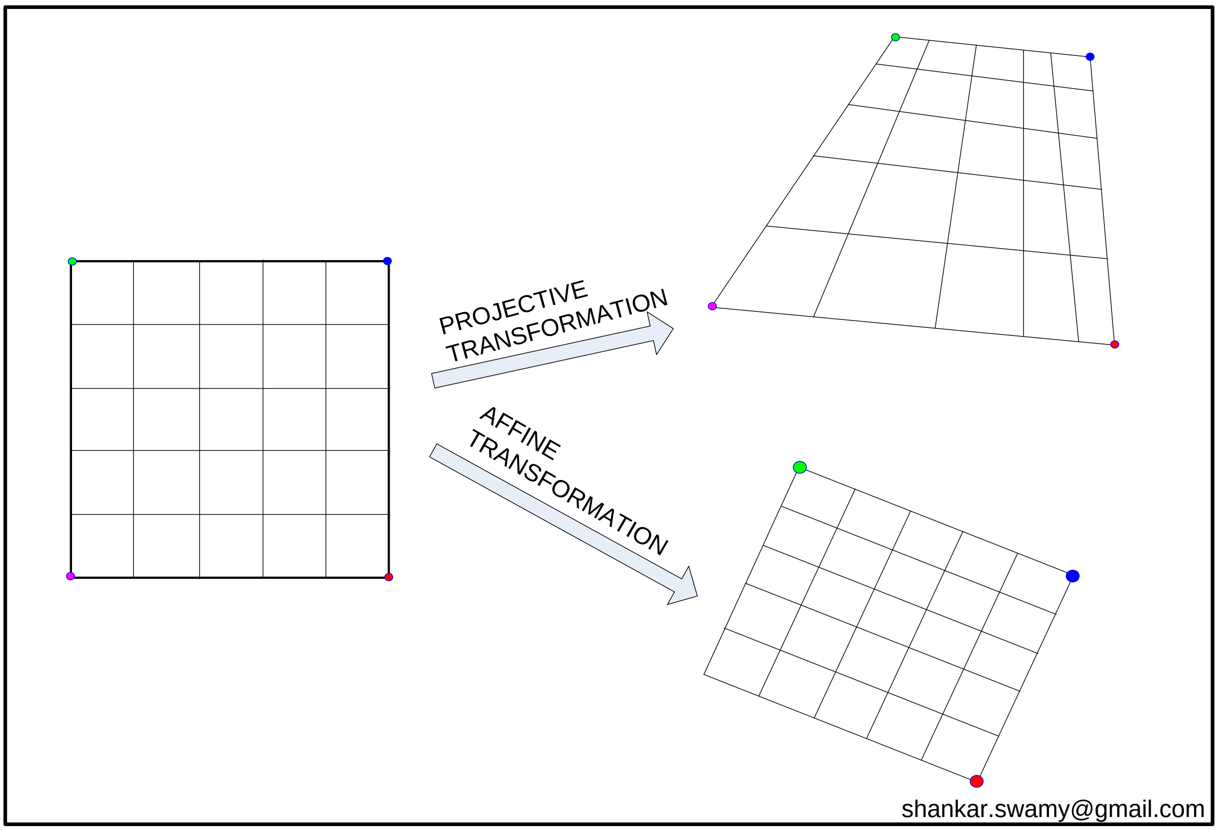

Figure 8-2 Projection and Affine Transformations illustrated

Table 1 Variant and Invariant properties of projective and affine transformations

Transformation Type	**Preserved property**	**Property Not Preserved**
Projective	a. Collinearity b. Incidence – two lines intersecting will continue to be so.	Parallelism Lengths Angle Ratios of distances
Affine	a. Collinearity – all points on a line will continue to be on a line b. Incidence **c. Parallelism** **d. Ratios of distances**	a. Angles b. Lengths

2.1.3 Notes:

a. That the affine transformations do not preserve the angles or the lengths means that any triangle can be transformed into any other triangle by an affine transformation. Hence the statement “all triangles are affine”.

b. Affine transformation, in general, is a combination of rotations, translations, dilations and shears.

2.1.4 Dilation

sdfasfafdsf

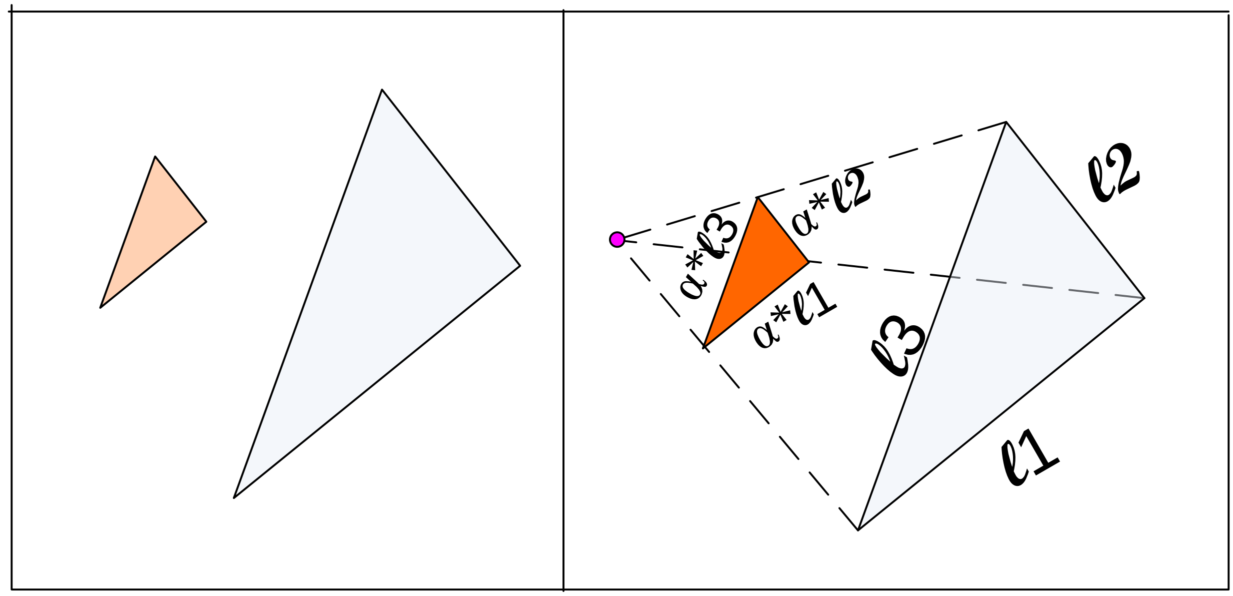

Figure 8-3 Illustration of a dilation transformation

In a dilation transformation of a geometric figure, each side of the figure is transformed by a constant multiple of the length. In the Figure above, that constant multiple is α.

2.1.5 Euclidian Transformation

This is either a translation or a rotation or a reflection.

This is a proper subset of the set of affine transformations. For example, in an Euclidian transformation a circle can only transform into another circle; but an affine transformation can transform a circle into an ellipse.

2.1.6 Otrhogonal Transformation

These are the transformations that can be represented by matrices which are orthogonal, that is, the matrices have their transposes as their inverses. Rotations are the only orthogonal transformation of interest in graphics.

Rotation and reflection combined results in an orthogonal transformation; but that is generally not encountered in graphics.

2.1.7 Rigid Transformation

These are transformations which represent the movement of solid objects – translations and rotations.

2.1.8 Similarity Transformations

Similarity transformation has the property that its transformation matrix T can be expressed in the format $T = P\,T'\,P^{-1}$

These are of significance in the study of fractals and other iterated function systems; but not encountered in 3D graphics.

2.2 The table that goes here is based on the one at:

http://www.euclideanspace.com/maths/geometry/affine/

2.2.1 Shear

The diagram below shows the shearing of a rectangle in the xy-plane along the y=0 line as the axis of shearing. In this transformation the points on the axis of shearing remains static. All other points move along the lines parallel to the axis of shearing. The amount by which they move is proportional to the distance of the points from the axis of shearing.

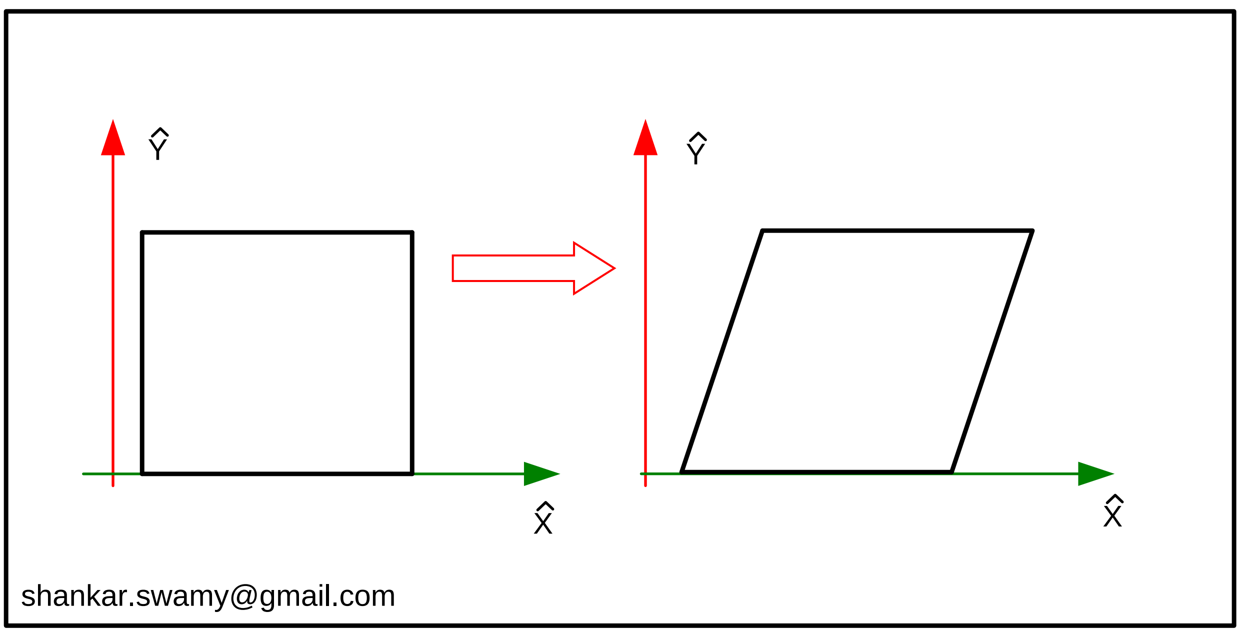

Figure 8-4 Shearing along y=0 axis

For such a shearing, the transformation is $x' = x + \lambda y\,; y' = y; z' = z$, or:

$$Sh_x = \begin{bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shearing a plane figure does not change its area. Shearing can be generalized to three dimensions where, instead of lines, planes are translated.

2.3 Transformations & Eigen Vectors

A nonzero vector is said to be an eigenvector of a linear transformation if the vector does not change its direction on application of the transformation. In such cases, the transformation can only change the length of the eigenvector. The amount by which the transformation changes the length of the vector is called the eigen value of the eigenvector for the transformation.

Not all transformations (that is, not all matrices) have eigenvectors. One extreme is a rotation matrix which has no non-trivial eigenvectors as the matrix leaves no vector of nonzero length unchanged in direction. The trivial eigenvectors of a rotation matrix are the vectors which lie on the axis of the rotation. On the other extreme is a scaling matrix with all vectors as eigenvectors – by definition a scaling matrix does not change the directions of the vectors it acts on.

In between these extremes is the shearing matrix which has some eigenvectors. Sheering by definition means all points along the selected line – axis of shearing - remain fixed while other points are shifted along the direction of the axis of shearing by an amount proportional to the distance of the points from the axis. Thus any vector that is parallel to the axis is an eigenvector of the transformation while any other vector is not an eigenvector. The eigen value of all eigenvectors of a shearing matrix is always one as shearing does not scale.