0.1 Continuous Functions on Intervals

Definition 0.1.1. *A function $f : A \to \mathbb{R}$ is said to be bounded on A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.*

Remark 0.1.2. *A function is bounded if the range of the function is a bounded set of $\mathbb{R}$. A continuous function is not necessarily bounded. For example, $f(x) = 1/x$ with $A = (0, \infty)$. But it is bounded on $[1, \infty)$.*

Theorem 0.1.3. *Let $I = [a, b]$ be a closed bounded interval, and $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

Proof. Suppose that f is not bounded on I. Then for each $n \in \mathbb{N}$, there exists $x_n \in I$ such that $|f(x_n)| > n$. As $x_n \in I$, so $\{x_n\}$ is bounded, by Bolzano-Weierstrass Theorem, there exists an accumulation point of $\{x_n\}$, so there exists a subsequence of $\{x_{n_k}\}$ so that $x_{n_k} \to x$. Since $a \leq x_{n_k} \leq b$, we also have $a \leq x \leq b$., i.e., $x \in I$. Since f is continuous on I, we must have $f(xn_k) \to f(x)$. But this is a contradiction since $|f(x_{n_k}| > n_k \geq k, k \in \mathbb{N}$. □

Remark 0.1.4. *In the proof, we use the result of earlier homework: if $x_n \leq b$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq b$. Similarly, if $x_n \geq a$ for all n, then $\lim_{n\to\infty} x_n \geq a$.*

Suppose that $f : S \to \mathbb{R}$. Then define the supremum of f on S, denoted $\sup_S f$, to be

$$\sup_S = \sup\{f(x) : x \in S\},$$

similarly, the infimum of f on S is defined by

$$\inf_S f = \inf\{f(x) : x \in\}.$$

Note that $\sup_S f$ could be ∞ and $\inf_S f$ could be $-\infty$, depending on the function f and S. For example, $f(x) = x^2$ and $S = \mathbb{R}$. Then $\sup_S f = +\infty$ and $\inf_S f = 0$. Now if $S = (0, 2)$, then $\sup_S f = 4$ and $\inf_S f = 0$. There are no points in $(0, 2)$ where f takes 0 and 4.

Remark 0.1.5. *If there is $c \in S$ such that $sup_S f = f(c)$, then f has an absolute maximum on S at c. Similarly for absolute minimum.*

Theorem 0.1.6. *(Maximau-Minimum Theorem) Let) $= [a, b]$ be a closed ad bounded interval, and $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and minimum, i.e., there exist points $c, d \in I$ such that*

$$f(c) = \sup_I f(x) \text{ and } f(d) = \inf_I f.$$

Proof. As f is continuous on I, so it is bounded. Hence both $\inf f$ and $\sup f$ exist. Thus, for each $n \in \mathbb{N}$, there is $x_n \in I$ such that

$$\sup_I f - \frac{1}{n} < f(x_n) \le \sup_I f.$$

So we have a sequence $\{x_n\} \subset I$, by Bolzana-Weierstrass, there exists a subsequence $x_{n_k} \to c$, so $f(x_{n_k}) \to f(c)$. But the limit is unique. Hence $f(c) = \sup_I f$. Similarly we can prove that $f(d) = \inf_I f$. □

Theorem 0.1.7. *(Bolzano's Intermediate Value Theorem) Let f be a continuous function on $[a, b]$ such that $f(a) \neq f(b)$. Let y be any real number between $f(a)$ and $f(b)$. Then there is a $c \in (a, b)$ such that $f(c) = y$.*

Proof. Without loss of generality, consider $f(a) < y < f(b)$. First define a set

$$S = \{x \in [a, b] : f(x) < y\}.$$

Thus, $S \neq \emptyset$ as $a \in S$. It is clear that S is bounded. So $\sup S$ exists, let $c = \sup S$. Now we prove that $f(c) = y$. It is clear that $a \le c \le b$.

Suppose that $c = a$. As $a \in S$ and f is continuous at a, so $f(a) < y$ which implies there exists a $\delta > 0$ such that $\forall x \in [a, a + \delta) \implies f(x) < y$. Each point in this neighborhood is in S. This is a contradiction to $c = \sup S$.

As $c = \sup S$, there exists a sequence $\{x_n\}$ in S such that $x_n \to c$. From $f(x_n) < y \implies f(c) \le y$, as f is continuous at c.

If $f(c) < y$, again by f being continuous at c, there is a neighborhood of c, $(c - \delta, c + \delta)$, such that $f(x) < y$ for all $x \in (c - \delta, c + \delta)$, which is contradiction, as $c = \sup S$. □

Example 0.1.8. *Consider $f(x) = x^2 - 2$ on $[0, 2]$. So $f(0) = -2, f(2) = 2$. Let $y = 0$. Then from the theorem, there exists $c \in (0, 2)$ such that $f(c) = 0$, in fact, $c = \sqrt{2}$.*

One note about this: if we only consider the set of rationals, then the graph of $x^2 - 2$ would cross the x-axis without meeting it. Another example of the set of real numbers complete (axiom 12).

Corollary 0.1.9. *Let f be continuous function on $[a,b]$ and define $m = \inf_I f$ and $M = \sup_I f$. Then the range of f is the interval $[m,M]$, i.e., $f([a,b]) = [m,M]$.*

Proof. We know from above, there exists $c, d \in [a,b]$ such that $f(c) = m, f(d) = M$. And any number $y \in (m,M)$, there is $c \in (a,b)$ such that $f(c) = y$. From the definition m, M, f does not have values outside $[m,M]$. So the range equals $[m,M]$. □

Theorem 0.1.10. *Let I be an interval and let $f : I\mathbb{R}$ be continuous on I. If $\alpha < \beta$ are numbers in I such that $f(\alpha) < 0 < f(\beta)$ (or $f(\alpha) > 0 > f(\beta)$), then there exists a number $c \in (\alpha, \beta)$ such that $f(c) = 0$.*

Proof. As f is continuous on I, so f is continuous on $[\alpha, \beta]$. Apply the Intermediate Value Theorem with $y = 0$. □

Example 0.1.11. *Let $f(x) = \frac{1}{x^2+1}$.*

1. $I_1 = (-1, 1)$. $f(I_1) = (\frac{1}{2}, 1]$.

2. $I_2 = [0, \infty)$, $f(I_2) = (0, 1]$.

Lemma 0.1.12. *Let $S \subseteq \mathbb{R}$ be a nonempty set with the property if $x, y \in S$ with $x < y$, then $[x, y] \subseteq S$. Then S is an interval.*

Theorem 0.1.13. *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then $f(I)$ is an interval.*

0.2 Uniform Continuity

First recall the definition of f being continuous at x_0: $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x : |x - x_0| < \epsilon \implies |f(x) - f(x_0)| < \epsilon$.

In general, δ depends on both ϵ and x_0, as function changes rapidly at some points and flat at some other points. We start some examples to look into this.

Example 0.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 2x$. Let $x_0 \in \mathbb{R}$. Consider*

$$|f(x) - f(x_0)| = |2x - 2x_0| = 2|x - x_0|.$$

From this we can see if we choose $\delta = \epsilon/2$, we have $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. In this case, δ depends only on ϵ, it works for all $x_0 \in \mathbb{R}$.

Example 0.2.2. *Let* $f : (0, \infty) \to \mathbb{R}$ *with* $f(x) = 1/x$*. Let* $x_0 = u > 0$*. Consider*

$$|f(x) - f(u)| = \frac{|x - u|}{xu}.$$

As $x \to u$*, so consider only* $|x - u| < u/2$*, i.e.,* $u/2 < x < 3u/2$*. Then* $1/x < 2/u$*. Hence* $1/ux < (1/u)(2/u) = 2/u^2$*. Now given* $\epsilon > 0$*, choose* $\delta = \min\{u/2, u^2\epsilon/2\}$*. So when* $|x - u| < \delta \implies |f(x) - f(u)| < \epsilon$*.*

Here δ *depends on both* ϵ *and* u*. In fact, there is no* δ *for all* $u > 0$*, as then* $\delta = 0$*.*

See the graph of $f(x) = 1/x$*.*

Definition 0.2.3. *Let* $f : D \to \mathbb{R}$ *is uniformly continuous on* $E \subset D$ *iff* $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x, y \in E, |x - y|, \delta \implies |f(x) - f(y)| < \epsilon$*. If* f *is uniformly continuous on* D*, then* f *is uniformly continuous.*

Remark 0.2.4. f *uniformly continuous on* E *implies* f *is continuous on* E*. The converse is not true.*

Example 0.2.5.

1. $f : [2.5, 3] \to \mathbb{R}$ *defined by* $f(x) = \frac{3}{x-2}$*.*
2. $f : (0, 6) \to \mathbb{R}$ *with* $f(x) = x^2 + 2x - 5$*.*
3. $f : (2, 3) \to \mathbb{R}$ *with* $f(x) = \frac{3}{x-2}$*.*

Non-uniform Continuity Criteriia Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent.

1. f is not uniformly continuous on A.
2. $\exists \epsilon_0 > 0$ such that for every $\delta > 0$ there are points $x_\delta, y_\delta \in A$ such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.
3. $\exists \epsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in A such that $\lim(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Example 0.2.6. *Let* $f : (0, \infty) \to \mathbb{R}$ *with* $f(x) = 1/x$*. Now pick* $\epsilon_0 = 1/2$*, and choose* $x_n = 1/n$ *and* $y_n = 1/(n + 1)$*. Then* $\lim(x_n - y_n) = 0$ *and* $|f(x_n) - f(y_n)| = 1 > 1/2$ *for all* n*.*

Theorem 0.2.7. *Let* I *be a closed bounded interval and let* $f : I \to \mathbb{R}$ *be continuous on* I*. Then* f *is uniformly continuous on* I*.*

Proof. If f is not uniformly continuous on I. From the above, $\exists \epsilon_0 > 0$ and $x_n, y_n \in I$ such that $x_n - y_n \to 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all n. As I is bounded, so by Bolzana-Weierstrass, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to $z \in I$, as I is closed interval. In addition, from

$$|y_{n_k} - z| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$y_{n_k} \to z$ as well.

Now as f is continuous at z, so we have $f(x_{n_k}) \to f(z)$ and $f(y_{n_k}) \to f(z)$. But this is a contradiction, as $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all n. □

Lipschitz Functions

Definition 0.2.8. *Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that*

$$|f(x) - f(y)| \leq K|x - y|, \ \forall x, y \in A,$$

then f is said to be a Lipschitz function (or to satisfy a Lipschitz condition) on A.

Geometrically, f is Lipschitz if and only if the slopes of secant line joining points $(x, f(x))$ and $(y, f(y))$ are bounded by K.

Theorem 0.2.9. *Let $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Proof. Let $\epsilon > 0$, choose $\delta = \epsilon/K$. Then for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. □

Example 0.2.10. *Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. f is uniformly continuous on $[a, b]$ but not on $\mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$. Consider

$$|f(x) - f(c)| = |x^2 - c^2| = |x - c||x + c|.$$

As x is close to c, we assume that $|x - c| < 1$. So this implies $|x| < 1 + |c|$, thus $|x + c| \leq 1 + 2|c|$. Hence

$$|x - c||x + c| < |x - c|(1 + 2|c|).$$

Now for $\epsilon > 0$, choose $\delta = \min\{1, \frac{\epsilon}{1+2|c|}\}$ such that for all x satisfying $|x-c| < \delta$, $|f(x) - f(c)| < \epsilon$, i.e., f is continuous on $\mathbb{R}$.

As δ depends on both ϵ and c, c is larger and larger, the values of δ is smaller and smaller (as the graph becomes more steeper). There is no such δ that works for all points. In fact, $\inf_c \delta = 0$.

But when we consider only on $[-a, a]$ for $a > 0$. Then

$$|x + c| \leq |x| + |c| \leq 2a,$$

hence $\delta = \epsilon/2a$ works for all points on $[-a, a]$, i.e., f is uniformly continuous on $[-a, a]$.

$\square$