Subelliptic Peter–Weyl and Plancherel theorems on compact, connected, semisimple Lie groups

András Domokos
Department of Mathematics and Statistics, California State University at Sacramento, 6000 J Street, Sacramento, CA, 95819, USA

A R T I C L E  I N F O
Article history:
Received 3 December 2014
Accepted 1 April 2015
Communicated by Enzo Mitidieri

K e y w o r d s:
Semisimple Lie groups
Sub-Riemannian geometry
Subelliptic Laplacian

A B S T R A C T
We will study the connections between the elliptic and subelliptic versions of the Peter–Weyl and Plancherel theorems, in the case when the sub-Riemannian structure is generated naturally by the choice of a Cartan subalgebra. Along the way we will introduce and study the subelliptic Casimir operator associated to the subelliptic Laplacian.

1. Introduction

In this paper we will show that the subelliptic spectral analysis results needed to study the subelliptic heat kernel and to obtain precise bounds of the $L^2$-norms of the non-horizontal directional derivatives, have natural and simple expressions in the case of compact and semi-simple Lie groups. The key ingredient is the connection between the elliptic and subelliptic versions of the Peter–Weyl and Plancherel theorems, when the subelliptic structure is generated by the orthogonal complement of a maximal commutative subalgebra of the Lie algebra.

Studies regarding the subelliptic heat kernel started immediately after Hörmander’s paper [13] appeared. Initially these studies were focused mostly on the nilpotent Lie groups [12], with a more recent shift toward non-nilpotent Lie groups [1–5], followed independently, and somewhat logically, by parallel studies on regularity of nonlinear subelliptic PDE’s [7,8,6].

Let $G$ be a compact, connected, semi-simple matrix Lie group and $\mathbb{G}$ its Lie algebra. In this context, working with matrix groups is not a restriction, but has the advantage of an easy setup for the formulas we will use. We denote by $\mathcal{M}_n(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{C})$ the vector space of $n \times n$ real or complex matrices and by $\text{GL}_n(\mathbb{R})$ or $\text{GL}_n(\mathbb{C})$ the Lie groups of the invertible $n \times n$ matrices. Matrix groups are defined as closed subgroups of $\text{GL}_n(\mathbb{R})$ or $\text{GL}_n(\mathbb{C})$ and hence inherit the Lie group structure.

E-mail address: domokos@csus.edu.
The Lie algebra of \( G \) can be defined by using the matrix exponential:

\[
G = \{ X \in M_n(\mathbb{R} \text{ or } \mathbb{C} \text{ depending on } G) \mid \exp(tX) \in G, \ \forall t \in \mathbb{R} \}.
\]

From this definition it follows that \( G \) is a real vector space and contains the generators of the 1-parameter subgroups of \( G \). \( G \) becomes an algebra when it is endowed with the bilinear operator,

\[
[X,Y] = XY - YX,
\]
called the commutator of \( X \) and \( Y \), which measures the non-commutativity at infinitesimal scales in \( G \). A commutative Lie algebra has the property that \([X,Y] = 0, \ \forall X,Y \in G\) and a semi-simple Lie algebra is on the opposite end of the commutativity scale, as it cannot have any non-trivial commutative ideals. A Lie group is semi-simple if its Lie algebra is semi-simple.

The adjoint representation of \( G \) is the group homomorphism

\[
\text{Ad} : G \to \text{Aut}(G), \quad \text{Ad}_x(X) = xXx^{-1},
\]
while its differential at the identity is the Lie algebra homomorphism

\[
\text{ad} : G \to \text{End}(G), \quad \text{ad}_X(Y) = [X,Y].
\]

The Killing form

\[
K(X,Y) = \text{trace}(\text{ad}_X \cdot \text{ad}_Y),
\]
is negative definite and non-degenerate on the Lie algebra of a compact, semi-simple Lie group, and hence we can define an inner product on \( G \) as

\[
\langle X,Y \rangle = -\rho K(X,Y), \quad (1.1)
\]
where \( \rho > 0 \) is a constant, which can be adjusted for each Lie algebra according to our normalization needs. The Killing form is \( \text{Ad} \) invariant, therefore \( \text{Ad}_x \) is a unitary linear transformation for all \( x \in G \) and \( \text{ad}_X \) is skew-symmetric for all \( X \in G \).

We will consider a natural sub-Riemannian geometry on \( G \), which is defined by the choice of a maximal, commutative subalgebra of \( G \), called a Cartan subalgebra.

Let us fix a Cartan subalgebra \( \Gamma \) and let \( T = \{T_1, \ldots, T_r\} \) be an orthonormal basis of it.

We extend the inner product of \( G \) bi-linearly to the complexified Lie algebra \( G_\mathbb{C} = G \oplus iG \). The mappings \( \text{ad}T : G_\mathbb{C} \to G_\mathbb{C}, \ T \in \Gamma \), commute and are skew-symmetric, so they can be simultaneously diagonalized and have purely imaginary eigenvalues.

We define \( \alpha \in \Gamma \) to be a root if \( \alpha \neq 0 \) and \( G_\alpha \neq \{0\} \), where

\[
G_\alpha = \{ Z \in G_\mathbb{C} : \text{ad}T(Z) = i \langle \alpha, T \rangle Z, \ \forall T \in T \}.
\]

Let \( \mathcal{R} \) be the set of all roots, which will be ordered by the relation \( \alpha > \beta \) if \( \alpha - \beta \) has its first non-zero coordinate positive. We denote by \( \mathcal{P} \) the set of all positive roots and let

\[
\delta = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} \alpha.
\]

For the most important properties of \( G_\alpha \) we quote [9,14]:

(i) \( \dim_\mathbb{C} G_\alpha = 1. \)
(ii) \( G_0 = \Gamma_\mathbb{C}. \)
(iii) \( G_{-\alpha} = \overline{G_\alpha}. \)
(iv) \( \langle G_\alpha, G_\beta \rangle = 0 \quad \text{if } \beta \neq \pm \alpha. \)
(v) $[G_\alpha, G_\beta] = \begin{cases} G_{\alpha + \beta} & \text{if } \alpha + \beta \in \mathcal{R} \\ \{0\} & \text{if } \alpha + \beta \notin \mathcal{R} \text{ and } \alpha + \beta \neq 0 \\ \subset i T & \text{if } \alpha + \beta = 0. \end{cases}$

(vi) If $E_\alpha \in G_\alpha$ and $E_{-\alpha} \in G_{-\alpha}$ then $[E_\alpha, E_{-\alpha}] = i \langle E_\alpha, E_{-\alpha} \rangle \alpha$.

The above properties of $G_\alpha$ and the real root space decomposition

$$G = T \oplus \bigoplus_{\alpha \in \mathcal{P}} \text{Re } (G_\alpha \oplus G_{-\alpha}),$$

allow us to choose an orthonormal basis of $\Gamma^\perp$,

$$\mathcal{X} = \{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}, \quad (1.3)$$

with the following properties:

**Proposition 1.1.**

(i) For all $1 \leq j \leq k$ there exists $\alpha_j \in \mathcal{P}$ such that $\{X_j, Y_j\} \in \text{Re}(G_{\alpha_j} \oplus G_{-\alpha_j})$

(ii) $E_{\pm\alpha_j} = X_j \pm Y_j \in G_{\pm\alpha_j}$.

(iii) $\langle E_{\alpha_j}, E_{-\alpha_j} \rangle = 2$.

(iv) $[X_j, Y_j] = -\alpha_j$.

(v) $\Gamma \subset [\mathcal{X}, \mathcal{X}]$.

The formula

$$Xf(x) = \left. \frac{d}{dt} \right|_{t=0} f(x \exp tx) \quad (1.4)$$

identifies each $X \in G$ with a left invariant vector field acting on smooth functions. By property (v) of **Proposition 1.1**, $\Gamma^\perp$ defines the horizontal distribution of a sub-Riemannian geometry on $G$. The subelliptic Laplacian corresponding to this sub-Riemannian geometry is

$$\Delta_{\mathcal{X}} = \sum_{j=1}^{k} X_j^2 + Y_j^2,$$

while the elliptic Laplacian is

$$\Delta = \sum_{j=1}^{k} X_j^2 + Y_j^2 + \sum_{m=1}^{r} T_m^2.$$

**Example 1.1 ([6,14]).** An example for a compact, semi-simple Lie group is the special orthogonal group

$$G = \text{SO}(n) = \{x \in \text{GL}_n(\mathbb{R}) : xx^T = I, \det x = 1\},$$

which has the Lie algebra,

$$G = \text{so}(n) = \{X \in \mathcal{M}_n(\mathbb{R}) : X + X^T = 0\}.$$

A basis of $\text{so}(n)$ is given by the matrices $X_{ij}$ for all $1 \leq i < j \leq n$, having all entries 0 except the $ij$th entry which is 1 and the $ji$th entry which is $-1$.

In $\text{so}(3)$ we can use $\rho = \frac{1}{2}$ in the inner product (1.1) and choose

$$T = X_{12}, \quad \alpha = 1, \quad X_1 = X_{13}, \quad Y_1 = X_{23}.$$
In so(4) we can use $\rho = \frac{1}{4}$ in the inner product (1.1) and choose

\[ T_1 = X_{12}, \quad T_2 = X_{34}, \]
\[ \alpha_1 = X_{12} - X_{34}, \quad \alpha_2 = X_{12} + X_{34}, \]
\[ X_1 = \frac{1}{\sqrt{2}} (X_{13} + X_{24}), \quad Y_1 = \frac{1}{\sqrt{2}} (X_{23} - X_{14}), \]
\[ X_2 = \frac{1}{\sqrt{2}} (X_{13} - X_{24}), \quad Y_2 = \frac{1}{\sqrt{2}} (X_{23} + X_{14}). \]

2. The subelliptic Peter–Weyl and Plancherel theorems

In this section we prove the subelliptic versions of the classical Peter–Weyl and Plancherel theorems, which are related to the spectral analysis of the subelliptic Laplacian.

Let $\hat{G}$ be the set of equivalence classes of irreducible, unitary representations of $G$. If $[\pi] \in \hat{G}$, then $\pi : G \to \text{Aut}V^\pi$ is a group homomorphism and $V^\pi$ is a finite dimensional complex vector space of dimension $d^\pi$. The derived representation of $\pi$ is defined as

\[ d\pi : G \to \text{End}(V^\pi), \quad d\pi(X)v = \frac{d}{dt} \bigg|_{t=0} \pi(e^{tX})v. \]

$d\pi$ is a Lie algebra homomorphism, as the differential at the identity of a smooth group homomorphism.

The entry functions of the representation $\pi$ are defined for every $u, v \in V^\pi$ by

\[ \pi_{u,v} : G \to \mathbb{C}, \quad \pi_{u,v}(x) = \langle \pi(x)u, v \rangle. \]

In our case of a compact, semi-simple Lie group, these entry functions are smooth and their directional derivatives in the directions of the flows of left invariant vector fields are calculated by

\[ X\pi_{u,v}(x) = \frac{d}{dt} \bigg|_{t=0} \pi_{u,v}(xe^{tX}) = \frac{d}{dt} \bigg|_{t=0} \langle \pi(xe^{tX})u, v \rangle \]
\[ = \frac{d}{dt} \bigg|_{t=0} \langle \pi(x) \circ \pi(e^{tX})u, v \rangle = \langle \pi(x) \circ d\pi(X)u, v \rangle. \quad (2.1) \]

As $G$ is compact, we can fix a normalized bi-invariant Haar measure $\mu$, and the entry functions $\pi_{u,v}$ belong to $L^2(G)$. Therefore, we can define the subspace of $L^2(G)$ generated by the entry functions of $\pi$,

\[ E^\pi = \text{span} \{ \pi_{u,v} : u, v \in V^\pi \} \subset L^2(G), \]

which depends only on the equivalence class of $\pi$.

We recall now Schur’s orthogonality relations [10,11]:

**Theorem 2.1.** Let $\pi$ be an irreducible, unitary representation of $G$. Then for all $u_1, u_2, v_1, v_2 \in V^\pi$ we have

\[ \int_G \pi_{u_1,v_1}(x) \cdot \overline{\pi_{u_2,v_2}(x)} d\mu(x) = \frac{1}{d^\pi} \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle. \quad (2.2) \]

**Theorem 2.2.** Let $\pi^1$ and $\pi^2$ be irreducible, unitary representations of $G$ such that $[\pi^1] \neq [\pi^2]$. Then for all $u_1, v_1 \in V^{\pi^1}$ and $u_2, v_2 \in V^{\pi^2}$ we have

\[ \int_G \pi_{u_1,v_1}^1(x) \cdot \overline{\pi_{u_2,v_2}^2(x)} d\mu(x) = 0. \quad (2.3) \]

The following theorem is fundamental for harmonic analysis on compact groups:
Theorem 2.3 (The Peter–Weyl Theorem).

\[ L^2(G) = \bigoplus_{[\pi] \in \hat{G}} \mathcal{E}_\pi, \]  

(2.4)

where \( \bigoplus \) means the closure of all finite linear combinations.

Corollary 2.1. If we choose an orthonormal basis \( \{e_\pi^j, 1 \leq j \leq d_\pi\} \) of each \( V_\pi \) and define \( \pi_{jk}(x) = (\pi(x)e_\pi^j, e_\pi^k) \), then

\[ \{-\sqrt{d_\pi} \pi_{jk} : [\pi] \in \hat{G}, 1 \leq j, k \leq d_\pi\} \]

is a complete orthonormal set in \( L^2(G) \) and therefore for all \( f \in L^2(G) \)

\[ f(x) = \sum_{[\pi] \in \hat{G}} \sum_{j,k=1}^{d_\pi} c_{jk}^\pi \pi_{jk}(x), \]  

(2.5)

where

\[ c_{jk}^\pi = d_\pi \int_G f(x) \overline{\pi_{jk}(x)} \, d\mu(x). \]

Definition 2.1. (a) The Casimir operator associated to the orthonormal basis \( \mathcal{X} \cup \mathcal{T} \) of \( G \) is

\[ \Omega^\pi : V^\pi \to V^\pi, \quad \Omega^\pi = \sum_{j=1}^{k} d\pi(X_j)^2 + d\pi(Y_j)^2 + \sum_{m=1}^{r} d\pi(T_m)^2. \]

(b) The subelliptic Casimir operator associated to \( \mathcal{X} \) is defined as

\[ \Omega^\pi_X : V^\pi \to V^\pi, \quad \Omega^\pi_X = \sum_{j=1}^{k} d\pi(X_j)^2 + d\pi(Y_j)^2. \]

We can observe that, as in the case of the adjoint representation, the linear mappings \( d\pi(T), \ T \in \Gamma \), commute and are skew symmetric, therefore they share eigenspaces and have purely imaginary eigenvalues. Hence, for \( \lambda \in \Gamma \) we can define

\[ V^\pi_{\lambda} = \{ v \in V : d\pi(T)v = i(\lambda, T)v, \ \forall \ T \in \Gamma \}, \]

and call \( \lambda \in \Gamma \) a weight of the representation \( d\pi \) if \( V^\pi_{\lambda} \neq \{0\} \). Let us denote by \( \Lambda^\pi \) the set of weights. On \( \Lambda^\pi \) we use the same ordering as on \( \mathcal{R} \) and let \( w^\pi \) be the highest weight. The following proposition is a well-established fundamental result:

Proposition 2.1 ([16]). For all \( u, v \in V^\pi \) we have

\[ \Omega^\pi u = -c^2 u, \]

and

\[ \Delta^\pi_{u,v} = -c^2 \pi_{u,v}, \]

where

\[ c^2 = \|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle. \]
Proof. In order to see the connection between Proposition 2.1 and its subelliptic counterpart, Proposition 2.2, we shortly explain the main ideas of the proof. The first identity is a consequence of Schur’s lemma and it follows from the facts that $d\pi(X)$ is skew symmetric, $d\pi$ is irreducible and $\Omega^\pi$ commutes with $d\pi$.

The second identity can be proved by using (2.1) to get that for any $X \in G$ we have

$$X^2 \pi_{u,v}(x) = \langle \pi(x) \circ d\pi(X)^2 u, v \rangle,$$

and subsequently,

$$\Delta \pi_{u,v}(x) = \langle \pi(x) \circ \Omega^\pi u, v \rangle = -c^2 \pi_{u,v}(x).$$

To estimate $c$, notice that if $\alpha \in P$, and $E_\alpha \in G_\alpha$ then

$$d\pi(E_\alpha)(V^\pi_w) = \{0\}.$$ 

For all $u \in V^\pi_w$ and $v \in V^\pi$ we have

$$E_\alpha \pi_{u,v}(x) = \langle \pi(x) \circ d\pi(E_\alpha) u, v \rangle = 0,$$

and hence

$$\sum_{j=1}^k \left( X_j^2 + Y_j^2 \right) \pi_{u,v} = \frac{1}{2} \sum_{j=1}^k \left( \langle E_\alpha_j E_{-\alpha_j} + E_{-\alpha_j} E_\alpha_j \rangle \pi_{u,v} \right)$$

$$= \frac{1}{2} \sum_{j=1}^k \left[ E_{-\alpha_j} E_{\alpha_j} \right] \pi_{u,v} + 2E_{-\alpha_j} E_\alpha_j \pi_{u,v}$$

$$= \sum_{\alpha \in P} i\alpha \pi_{u,v} = \sum_{\alpha \in P} -\langle w^\pi, \alpha \rangle \pi_{u,v}$$

$$= -2\langle w^\pi, \delta \rangle \pi_{u,v}.$$ 

Moreover,

$$T_m^2 \pi_{u,v}(x) = \langle \pi(x) \circ d\pi(T_m)^2 u, v \rangle = -\langle w^\pi, T_m \rangle^2 \pi_{u,v}(x),$$

which implies that

$$\sum_{m=1}^r T_m^2 \pi_{u,v} = -\sum_{m=1}^r \langle w^\pi, T_m \rangle^2 \pi_{u,v} = -\| w^\pi \|^2 \pi_{u,v}.$$ 

In conclusion, for $u \in V^\pi_w$ and $v \in V^\pi$ we have

$$\Delta \pi_{u,v} = \sum_{j=1}^k \left( X_j^2 + Y_j^2 \right) \pi_{u,v} + \sum_{m=1}^r T_m^2 \pi_{u,v}$$

$$= -\| w^\pi \|^2 + 2\langle w^\pi, \delta \rangle \pi_{u,v},$$

which proves that $c^2 = \| w^\pi \|^2 + 2\langle w^\pi, \delta \rangle$. 

Proposition 2.1 shows that the decomposition of $L^2(G)$ in the Peter–Weyl theorem gives the spectral decomposition of the elliptic Laplacian.

We propose to get a similar result for the subelliptic Laplacian. As a first note, the subelliptic Casimir operator $\Omega^\pi_\lambda$ does not commute with $d\pi$. Associated to the eigenspaces $V^\pi_\lambda$ there is an orthogonal direct space decomposition

$$V^\pi = \bigoplus_{\lambda \in A^\pi} V^\pi_\lambda.$$ (2.6)
Proposition 2.2. For all $u \in V^\pi_\lambda$ and $v \in V^\pi$ we have
\[ \Omega^\pi_\lambda u = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle - \|\lambda\|^2\right) u, \]
and
\[ \Delta^\pi_{u,v} = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle - \|\lambda\|^2\right) \pi_{u,v}. \]

Proof. Let us consider $T \in \Gamma, u \in V^\pi_\lambda$ and $v \in V^\pi$. Then
\[ d\pi(T)^2 u = -\langle \lambda, T \rangle^2 u, \]
and hence
\[ \Omega^\pi_\lambda u = \Omega^\pi u - \sum_{m=1}^r d\pi(T_m)^2 u = \Omega^\pi u + \sum_{m=1}^r \langle \lambda, T_m \rangle^2 u \]
\[ = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle - \|\lambda\|^2\right) u. \]

Therefore, for all $x \in \mathbb{G}$ we have
\[ \Delta^\pi_{u,v}(x) = \langle \pi(x) \circ \Omega^\pi_\lambda u, v \rangle = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle - \|\lambda\|^2\right) \pi_{u,v}(x). \]

Similarly to $E^\pi = \text{span}\{\pi_{u,v} : u, v \in V^\pi\}$, for $\lambda \in \Lambda^\pi$ let us define
\[ E^\pi_\lambda = \text{span}\{\pi_{u,v} : u \in V^\pi_\lambda, v \in V^\pi\}. \]

As an immediate consequence of Propositions 2.1 and 2.2 we have the following corollary.

Corollary 2.2. (a) For all $f \in E^\pi$ we have
\[ \Delta f = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle\right) f. \]
(b) For all $f \in E^\pi_\lambda$ we have
\[ \Delta^\pi f = -\left(\|w^\pi\|^2 + 2\langle w^\pi, \delta \rangle - \|\lambda\|^2\right) f. \]

For each $V^\pi_\lambda, \lambda \in \Lambda^\pi$, let us choose on orthonormal basis $\{e^\lambda_j\}$ and define the functions
\[ \pi^\lambda_{j,k}(x) = \langle \pi(x) e^\lambda_j, e^\lambda_k \rangle. \]

Now, as Corollary 2.2 gives the eigenspaces of the subelliptic Laplacian, everything is prepared to state and prove the subelliptic version of Theorem 2.3. We use $d^\pi_\lambda$ to denote the dimension of $V^\pi_\lambda$.

Theorem 2.4 (The Subelliptic Peter–Weyl Theorem).
\[ L^2(\mathbb{G}) = \bigoplus_{|\pi| \in \mathbb{G}} \bigoplus_{\lambda \in \Lambda^\pi} E^\pi_\lambda, \]

and hence for all $f \in L^2(\mathbb{G})$ we have
\[ f(x) = \sum_{|\pi| \in \mathbb{G}} \sum_{\lambda \in \Lambda^\pi} d^\pi_\lambda \sum_{j,k=1}^{d^\pi_\lambda} \pi^\lambda_{j,k}(x), \]
where
\[ c_{jk}^{\pi,\lambda} = d^\pi \int_G f(x) \overline{\pi_{jk}^{\pi}(x)} \, d\mu(x). \]

**Proof.** Schur’s orthogonality relation (2.3) from Theorem 2.2 and the orthogonal decomposition (2.6) imply that
\[ \mathcal{E}^\pi = \bigoplus_{\lambda \in \Lambda^\pi} \mathcal{E}_\lambda^\pi, \]
which combined with (2.4) from the Peter–Weyl theorem gives (2.7). The subelliptic Fourier series (2.8) is a direct consequence of (2.7).

As an immediate application of the two versions of the Peter–Weyl theorem, consider the elliptic heat equation on \( G \),
\[ \frac{\partial f}{\partial t}(t, x) = \Delta f(t, x), \quad f(0, x) = g(x) \in L^2(G), \]
and the subelliptic heat equation on \( G \),
\[ \frac{\partial f}{\partial t}(t, x) = \Delta_X f(t, x), \quad f(0, x) = g(x) \in L^2(G). \]

Theorems 2.3 and 2.4 give the following corollary.

**Corollary 2.3.** (a) The solution of the elliptic heat equation has the series form
\[ f(t, x) = \sum_{[\pi] \in \hat{G}} \sum_{j,k=1}^{d^\pi} c_{jk}^{\pi} e^{-(\|w\|^2 + 2\langle w, \delta \rangle) t} \pi_{jk}(x), \]
where
\[ c_{jk}^{\pi} = d^\pi \int_G g(x) \overline{\pi_{jk}(x)} \, d\mu(x). \]

(b) The solution of the subelliptic heat equation has the series form
\[ f(t, x) = \sum_{[\pi] \in \hat{G}} \sum_{\lambda \in \Lambda^\pi} \sum_{j,k=1}^{d^\pi} c_{jk}^{\pi,\lambda} e^{-(\|w\|^2 + 2\langle w, \delta \rangle - \|\lambda\|^2) t} \pi_{jk}(x), \]
where
\[ c_{jk}^{\pi,\lambda} = d^\pi \int_G g(x) \overline{\pi_{jk}^{\pi,\lambda}(x)} \, d\mu(x). \]

In the following we propose to obtain forms of the Peter–Weyl theorems which are less dependent of the choice of the basis of \( V^{\pi} \). For this, we denote the group convolution by
\[ f_1 \ast f_2(y) = \int_G f_1(x) f_2(x^{-1} y) \, d\mu(x), \]
and the character of the representation \( \pi \) by
\[ \chi^{\pi}(x) = \text{trace} \, \pi(x). \]
Theorem 2.5 (The Plancherel Theorem [11]). For any $f \in L^2(\mathbb{G})$ we have

$$f(x) = \sum_{[\pi] \in \hat{\mathbb{G}}} d^\pi f \ast \chi^\pi(x),$$

(2.9)

and

$$\|f\|_{L^2(\mathbb{G})} = \sum_{[\pi] \in \hat{\mathbb{G}}} (d^\pi)^2 \|f \ast \chi^\pi\|_{L^2(\mathbb{G})}^2,$$

(2.10)

where $d^\pi f \ast \chi^\pi$ is the orthogonal projection of $f$ onto $\mathcal{E}^\pi$.

A compact group is unimodular, so we can use the following result:

Proposition 2.3 ([15]). If $f, \varphi, \psi \in L^2(\mathbb{G})$ then

$$\langle f \ast \varphi, \psi \rangle_{L^2(\mathbb{G})} = \langle f, \psi \ast \tilde{\varphi} \rangle_{L^2(\mathbb{G})} = \langle \varphi, \tilde{f} \ast \psi \rangle_{L^2(\mathbb{G})},$$

(2.11)

where $\tilde{f}(x) = \overline{f(x^{-1})}$.

To simplify our notations, in the following we drop the superscript $\lambda$ and denote by $\{e_j^\pi\}$ an orthonormal basis of $V^\pi$ which corresponds to the orthogonal direct space decomposition (2.6) and hence it is the union of orthonormal bases of all $V^\pi_\lambda, \lambda \in \Lambda^\pi$. Let us define the subelliptic characters as

$$\chi^\pi_\lambda = \sum_{e_j^\pi \in V^\pi_\lambda} \pi_{jj},$$

(2.12)

Therefore,

$$\chi^\pi = \sum_{\lambda \in \Lambda^\pi} \chi^\pi_\lambda,$$

(2.13)

and we can state the subelliptic form of Theorem 2.5.

Theorem 2.6 (The Subelliptic Plancherel Theorem). For any $f \in L^2(\mathbb{G})$ we have

$$f(x) = \sum_{[\pi] \in \hat{\mathbb{G}}} \sum_{\lambda \in \Lambda^\pi} d^\pi f \ast \chi^\pi_\lambda(x),$$

(2.14)

and

$$\|f\|_{L^2(\mathbb{G})} = \sum_{[\pi] \in \hat{\mathbb{G}}} \sum_{\lambda \in \Lambda^\pi} (d^\pi)^2 \|f \ast \chi^\pi_\lambda\|_{L^2(\mathbb{G})}^2,$$

(2.15)

where $d^\pi f \ast \chi^\pi_\lambda$ is the orthogonal projection of $f$ onto $\mathcal{E}^\pi_\lambda$.

Proof. First, (2.14) follows from (2.9) and (2.13).

Second, let us observe that by Theorem 2.1 and Proposition 2.3, if $j \neq l$ then

$$\pi_{jk} \ast \pi_{ll}(y) = \int_\mathbb{G} \pi_{jk}(x) \overline{\pi_{ll}(x^{-1}y)} \, d\mu(x) = \int_\mathbb{G} \pi_{jk}(x) \overline{\pi_{ll}(y^{-1}x)} \, d\mu(x)$$

$$= \int_\mathbb{G} \langle \pi(x) e_j^\pi, e_l^\pi \rangle \overline{\langle \pi(y) e_j^\pi, \pi(y) e_l^\pi \rangle} \, d\mu(x) = 0.$$

Hence,

$$\langle f \ast \pi_{ll}, f \ast \pi_{jj} \rangle_{L^2(\mathbb{G})} = \langle \pi_{ll}, \tilde{f} \ast f \ast \pi_{jj} \rangle_{L^2(\mathbb{G})} = \langle \pi_{ll} \ast \tilde{\pi}_{jj}, \tilde{f} \ast f \rangle_{L^2(\mathbb{G})} = 0,$$

(2.16)
and

$$\langle f \ast \pi_{ll}, \pi_{jk} \rangle_{L^2(G)} = \langle f, \pi_{jk} \ast \pi_{ll} \rangle_{L^2(G)} = 0. \tag{2.17}$$

By (2.16), if \( \lambda_1 \neq \lambda_2 \), then \( f \ast \chi_{\lambda_1} \) is orthogonal to \( f \ast \chi_{\lambda_2} \), which gives (2.15). Moreover, by (2.17), if \( \pi_{jj} \in \mathcal{E}_\lambda \), then \( f \ast \pi_{jj} \in \mathcal{E}_\lambda \) and therefore \( f \ast \chi_\lambda \) is the orthogonal projection of \( f \) onto \( \mathcal{E}_\lambda \). \qed

3. The case of SU(2)

Let us explore how the subelliptic versions of the Peter–Weyl and Plancherel theorems look in the case of the special unitary group

$$\text{SU}(2) = \left\{ x_{a,b} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$ 

Its Lie algebra is the three dimensional real vector space

$$\text{su}(2) = \left\{ X = \begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$ 

The Killing form of \( \text{su}(2) \) is

$$K(X, Y) = 4 \text{trace}(XY),$$

while the inner product (1.1) is defined as

$$\langle X, Y \rangle = -\frac{1}{2} \text{trace}(XY).$$

The Cartan subalgebra \( \Gamma \) is spanned by

$$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and the basis of the orthogonal complement of \( \Gamma \) is

$$X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$ 

For integers \( m \geq 0 \) consider the vector space of homogeneous polynomials of degree \( m \),

$$P_m = \left\{ P(z, v) = \sum_{j=0}^m c_j z^j v^{m-j} : c_0, \ldots, c_m \in \mathbb{C} \right\},$$

endowed with the inner product

$$\langle P, Q \rangle = \int_{S^3} P(z, v)\overline{Q(z, v)} \, d\sigma(z, v),$$

where \( d\sigma \) is the normalized surface measure on \( S^3 \). An orthonormal basis of \( P_m \) is formed by

$$P^m_j(z, v) = \alpha_j^m z^j v^{m-j}, \quad 0 \leq j \leq m,$$

where

$$\alpha_j^m = \sqrt{\frac{m!}{j!(m-j)!}}.$$
A unitary, irreducible representation of SU(2) onto $P_m$ can be defined as $\pi^m : SU(2) \to Aut(P_m)$,

$$\pi^m(x_{a,b})P(z,v) = P((z,v) \cdot x_{a,b}) = P(az - bv, bz + \bar{a}v).$$

It is a well-known result [10,11] that $SU(2) = \{[\pi^m] : m = 0, 1, 2, \ldots\}$, so we can start computing the eigenvalues and eigenvectors of the Casimir operators and Laplacians.

Continuing the calculations we get

$$d\pi^m(T)P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} \pi^m(e^{it})P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} P^m_j \left( (z,v) \cdot \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right) = i(2j - m)P^m_j(z,v).$$

Therefore, the weights of the representation $d\pi^m$ are $\lambda^m_j = (2j - m)T$ and the orthogonal direct space decomposition (2.6) is formed by the one dimensional subspaces

$$V_{\lambda^m_j} = \text{span}\{P^m_j\}.$$ 

Continuing the calculations we get

$$d\pi^m(X_1)P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} \pi^m(e^{itX_1})P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} P^m_j \left( (z,v) \cdot \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = \alpha^m_j \left( -jz^{j-1}v^{m-j+1} + (m - j)z^{j+1}v^{m-j-1} \right),$$

and hence

$$d\pi^m(X_1)^2P^m_j(z,v) = \alpha^m_j(j(j - 1)z^{j-2}v^{m-j+2} - j(m - j + 1)z^jv^{m-j} - (m - j)(j + 1)z^jv^{m-j} - (m - j)(m - j - 1)z^{j+2}v^{m-j-2}).$$

In a similar way,

$$d\pi^m(X_2)P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} \pi^m(e^{itX_2})P^m_j(z,v) = \left. \frac{d}{dt} \right|_{t=0} P^m_j \left( (z,v) \cdot \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} \right) = \alpha^m_j \left( iz^{j-1}v^{m-j+1} + i(m - j)z^{j+1}v^{m-j-1} \right),$$

and

$$d\pi^m(X_2)^2P^m_j(z,v) = \alpha^m_j(-j(j - 1)z^{j-2}v^{m-j+2} - j(m - j + 1)z^jv^{m-j} - (m - j)(j + 1)z^jv^{m-j} - (m - j)(m - j - 1)z^{j+2}v^{m-j-2}).$$

In conclusion,

$$\Omega^m_{\lambda^m_j} = -(4mj - 4j^2 + 2m)P^m_j,$$

and

$$\Omega^m P^m_j = -(m^2 + 2m)P^m_j.$$
The entry functions of the representation $\pi^m$ are

$$\pi^m_{jk}(x_{a,b}) = \langle \pi^m(x_{a,b})P^m_j, P^m_k \rangle = \int_{S^3} (az - \overline{bv})^j (bz - \overline{av})^{m-j} z^{m-k} d\sigma(z, v). \quad (3.3)$$

By Proposition 2.2

$$\Delta_X \pi^m_{jk} = -(4mj - 4j^2 + 2m)\pi^m_{jk}, \quad (3.4)$$

and

$$\Delta \pi^m_{jk} = -(m^2 + 2m)\pi^m_{jk}. \quad (3.5)$$

Hence, for all $f \in L^2(SU(2))$ we have

$$f(x_{a,b}) = \sum_{m=0}^{\infty} \sum_{j,k=0}^{m} c^m_{jk} \pi^m_{jk}(x_{a,b}), \quad (3.6)$$

which corresponds to both the elliptic and subelliptic Peter–Weyl theorems and implicitly to the spectral decomposition of both the elliptic and subelliptic Laplacian. Therefore, the solution of the elliptic heat equation in $SU(2)$ has the form

$$f(t, x_{a,b}) = \sum_{m=0}^{\infty} \sum_{j,k=0}^{m} c^m_{jk} e^{-(m^2+2m)t} \pi^m_{jk}(x_{a,b}),$$

while the solution of the subelliptic heat equation looks like

$$f(t, x_{a,b}) = \sum_{m=0}^{\infty} \sum_{j,k=0}^{m} c^m_{jk} e^{-(4mj-4j^2+2m)t} \pi^m_{jk}(x_{a,b}).$$

References