1. Give an example for each of the following

   (a) A convergent series in which the ratio test is inconclusive.

   (b) A divergent series in which the ratio test is inconclusive.

   (c) A divergent series, $\sum_{n=0}^{\infty} a_n$, such that $\lim_{n\to\infty} a_n = 0$.

   (d) A conditionally convergent series.

2. Consider the series $\sum_{n=1}^{\infty} \frac{3}{n^2}$. Find the third partial sum and approximate the error. Be sure to show all work.

$$s_3 = \frac{3}{(1)^2} + \frac{3}{(2)^2} + \frac{3}{(3)^2} = \frac{49}{12} \approx 4.083$$

$$\begin{aligned}
\int_4^\infty \frac{3}{x^2}\,dx &\le R_3 \le \int_3^\infty \frac{3}{x^2}\,dx \\
\lim_{t\to\infty}\int_4^t \frac{3}{x^2}\,dx &\le R_3 \le \lim_{t\to\infty}\int_3^t \frac{3}{x^2}\,dx \\
\lim_{t\to\infty} -\frac{3}{x}\bigg|_4^t &\le R_3 \le \lim_{t\to\infty} -\frac{3}{x}\bigg|_3^t \\
\lim_{t\to\infty} -\frac{3}{t} + \frac{3}{4} &\le R_3 \le \lim_{t\to\infty} -\frac{3}{t} + \frac{3}{3} \\
\frac{3}{4} &\le R_3 \le 1
\end{aligned}$$

The third partial sum is about 4.083 with an error of somewhere between $\frac{3}{4}$ and 1.

3. Determine whether $\sum_{n=1}^{\infty} \frac{n^2+n-1}{3n^3+n}$ converges or $\boxed{\text{diverges.}}$

I will use the Limit Comparison Test and compare to $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+n-1}{3n^3+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+n^2-n}{3n^3+n} = \frac{1}{3} > 0$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n^2+n-1}{3n^3+n}$ diverges.

4. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3-1}$ converges absolutely, $\boxed{\text{converges conditionally}}$ or diverges.

$$\left|\frac{(-1)^n n^2}{n^3-1}\right| = \frac{n^2}{n^3-1} > \frac{n^2}{n^3} = \frac{1}{n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know by Comparison Test that $\sum_{n=1}^{\infty} \left|\frac{(-1)^n n^2}{n^3-1}\right|$ diverges. Therefore the original series does not converge absolutely.

Now we will check to see if it converges conditionally, by using the Alternating Series Test.

$$\lim_{n\to\infty} \frac{n^2}{n^3-1} = 0$$

$$\begin{aligned} f(x) &= \frac{x^2}{x^3-1} \\ f'(x) &= \frac{(x^3-1)(2x) - x^2(3x^2)}{(x^3-1)^2} \\ &= \frac{-x^4-2x}{(x^3-1)^2} < 0 \text{ on } [1,\infty) \end{aligned}$$

Therefore $f(x)$ is decreasing and hence $b_{n+1} < b_n$. So by the Alternating Series Test, $\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{n^3-1}$ converges.

Since it did not converge when we took the absolute value, but did converge with the negatives still in, we know that it converges conditionally.

5. Determine whether $\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{n!}$ $\boxed{\text{converges absolutely}}$, converges conditionally or diverges.

$$\lim_{n\to\infty} \left|\frac{\frac{(-3)^{n+1}(n+1)^2}{(n+1)!}}{\frac{(-3)^n n^2}{n!}}\right| = \lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = 0$$

Therefore by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{n!}$ converges absolutely.

6. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n+1}$.

   (a) Where is this power series centered? $a = \frac{5}{2}$

   (b) Determine the interval of convergence.

$$\lim_{n\to\infty} \left| \frac{\frac{(2x-5)^{n+1}}{n+2}}{\frac{(2x-5)^n}{n+1}} \right| = \lim_{n\to\infty} \frac{|2x-5|^{n+1}}{n+2} \cdot \frac{n+1}{|2x-5|^n} = \lim_{n\to\infty} \frac{|2x-5|(n+1)}{n+2} = |2x-5|$$

$$\begin{aligned} |2x-5| &< 1 \\ -1 < 2x-5 &< 1 \\ 4 < 2x &< 6 \\ 2 < x &< 3 \end{aligned}$$

   Now we need to check the endpoints.

   Plug in $x = 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ which converges by alternating series test.

   Plug in $x = 3$, $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$ which diverges by comparison to $\frac{1}{n}$.

   Therefore the **interval of convergence is $[2, 3)$.**

   (c) What is the radius of convergence? $R = \frac{1}{2}$

7. Represent each of the following functions as a power series.

   (a) $f(x) = \dfrac{x}{1+x^6}$

$$\frac{x}{1+x^6} = \sum_{n=0} x(-x^6)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{6n+1}}$$

   (b) $f(x) = \dfrac{1}{(1-2x)^2}$

$$\begin{aligned} \frac{1}{(1-2x)^2} &= \frac{1}{2}\frac{d}{dx}\left(\frac{1}{1-2x}\right) \\ &= \frac{1}{2}\frac{d}{dx}\sum_{n=0}^{\infty}(2x)^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty} n(2x)^{n-1}2 \\ &= \boxed{\sum_{n=1}^{\infty} n(2x)^{n-1}} \end{aligned}$$

8. Use your answer to problem 7a to determine $\int_0^{.5} \frac{x}{1+x^6}$ correct to within .0001.

$$\begin{aligned}\int_0^{.5} \frac{x}{1+x^6} &= \int_0^{.5} \sum_{n=0}^{\infty} (-1)^n x^{6n+1}\ dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{6n+2}\Bigg|_0^{.5} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (.5)^{6n+2}}{6n+2} \\ &= \frac{.5^2}{2} - \frac{.5^8}{8} + \frac{.5^{14}}{14} - \frac{.5^{20}}{20} + \cdots\end{aligned}$$

Since $\frac{.5^{14}}{14} \approx .000004 < .0001$, we need only take the first two terms and this will give us an accurate enought estimation.

Therefore $\int_0^{.5} \frac{x}{1+x^6} \approx \frac{.5^2}{2} - \frac{.5^8}{8} \approx .1245.$