Chapter 17

Unmixedness and the Generalized Principal Ideal Theorem

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Abstract

Recent work toward extending the theory of Cohen-Macaulayness to all commutative rings (both Noetherian and non-Noetherian) has led to the definition of weak Bourbaki unmixed rings (wB-unmixed) and weak Bourbaki height-unmixed rings (wB-ht-unmixed). In this work we study these unmixedness conditions on rings which satisfy the generalized principal ideal theorem (GPIT) or at least the principal ideal theorem (PIT). This is a natural extension from Noetherian rings since Noetherian rings satisfy GPIT and PIT. There are, however, many rings which satisfy GPIT and/or PIT which are not Noetherian. Among the results are the following: (1) In rings which satisfy GPIT, wB-ht-unmixed is equivalent to wB-unmixed. (2) Every unmixed domain (in either sense) satisfies PIT. (3) Locally Cohen-Macaulay rings (which are locally Noetherian and therefore satisfy GPIT) are unmixed. As a corollary to result (2) we also get that a Prüfer domain R is wB-ht-unmixed if and only if $\dim(R) \leq 1$.

1 Introduction

In two recent papers ([1] and [2]) we have studied wB-ht-unmixed rings and wB-unmixed rings as candidates for an appropriate definition of non-Noetherian Cohen-Macaulayness. The question of whether there exists an appropriate definition of non-Noetherian Cohen-Macaulayness was first posed by Sarah Glaz in 1992 [3] and then again in 1994 [4]. WB-ht-unmixedness and wB-unmixedness (which will be defined in Section 2) have each been shown to satisfy, at least, some of the requirements for an appropriate definition of non-Noetherian Cohen-Macaulayness.

To determine whether either of these types of rings is an appropriate definition of non-Noetherian Cohen-Macaulayness we must learn as much as possible about these types of rings. As with most questions about non-Noetherian rings, it is difficult to proceed without placing further restrictions on the ring. In this paper we will consider these unmixedness questions on rings which satisfy either the principal ideal theorem (PIT) or the generalized principal ideal theorem (GPIT) conditions. We will find that in rings which satisfy GPIT, these two notions of unmixedness are equivalent. We will also find that unmixed (in either sense) domains must satisfy PIT. These results will allow us to completely classify Prüfer domains with respect to unmixedness.

2 Unmixedness

All rings are commutative rings with unity.

Recall that an ideal I is said to be height-unmixed if all the associated primes of I have equal height. That is $\mathrm{ht}(P) = \mathrm{ht}(Q)$ for all $P, Q \in$ all $\mathrm{Assoc}(I)$ where $\mathrm{Assoc}(I)$ denotes the set of associated prime ideals of I.

An ideal I is said to be unmixed if there are no embedded primes among the associated primes of I. That is, $P \subseteq Q \implies P = Q$ for all $P, Q \in \mathrm{Assoc}(I)$.

Lemma 2.1. *If R is a locally finite dimensional (LFD) ring, then all height-unmixed ideals are unmixed.*

Proof. Suppose I is a height-unmixed ideal in a LFD ring. Let $P, Q \in \mathrm{Assoc}(I)$ with $P \subseteq Q$.Then $\mathrm{ht}(P) = \mathrm{ht}(Q)$ since I is height-unmixed. Also, $\mathrm{ht}(Q) < \infty$since the ring is LFD. Therefore, since $P \subseteq Q$, we have $P = Q$. ∎

For a non-Noetherian ring, there are many types of associated primes which may be used in the above definitions of unmixedness. In 1984 Iroz and Rush [5] published a paper describing most of the commonly used sets of associated primes in non-Noetherian rings all of which are equivalent in a Noetherian ring. One of the most commonly studied sets of associated primes, and the one that will be used here, is the set of weak Bourbaki associated primes. The set of weak Bourbaki associated primes to an ideal I is denoted $\mathrm{Ass}_f(I)$. A prime ideal P is a weak Bourbaki associated prime of the ideal I if $P \in \mathrm{Min}(I : a)$ for some $a \in R$. Since $(I : a) = R$ if $a \in I$, we have

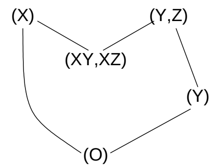
$$\mathrm{Ass}_f(I) = \bigcup_{a \in R} \mathrm{Min}\,(I : a) = \bigcup_{a \notin I} \mathrm{Min}\,(I : a)\,.$$

One reason for studying these primes, as opposed to some other set of associated primes, is that the set of weak Bourbaki associated primes of an ideal I is the smallest set of commonly studied associated primes which is necessarily non-empty for each proper ideal I.

Note: we will say that an ideal is wB-ht-unmixed if it is height-unmixed with respect to the set of weak Bourbaki associated primes and an ideal is wB-unmixed if it is unmixed with respect to the set of weak Bourbaki associated primes.

The following example shows that an unmixed ideal need not be height-unmixed even if the ring is LFD (in fact, even if the ring is Noetherian).

Example 2.2. *Let $R = k[X, Y, Z]$ where k is a field and let $I = (XY, XZ)$. Then both (X) and (Y, Z) are in $\mathrm{Ass}_f(I)$. These ideals have different heights, so $I$is not height-unmixed with respect to the weak Bourbaki associated primes. However, I has no embedded components with respect to the weak Bourbaki associated primes, so I is unmixed.*



We are almost ready for the definitions of wB-ht-unmixed and wB-unmixed rings. First, recall that in [1] a height-generated ideal was defined to be a finitely generated ideal I which can be generated by ht(I) elements. That is, a finitely generated ideal I is height-generated if $\mu(I) \leq$ht(I) where $\mu(I)$denotes the minimal number of generators of I. It is straightforward to show that in a Noetherian ring wB-unmixed is equivalent to wB-ht-unmixed for height-generated ideals, so, for height-generated ideals in Noetherian rings, we may simply refer to the ideal being unmixed to refer to either of the conditions wB-unmixed or wB-ht-unmixed.

To see why wB-ht-unmixed and wB-unmixed rings have been chosen as candidates for an appropriate definition of non-Noetherian Cohen-Macaulayness we first recall the definition of a Cohen-Macaulay ring.

Definition 2.3. *A Noetherian ring R Cohen-Macaulay if every height-generated ideal in R is unmixed.*

To extend this to an appropriate definition of non-Noetherian Cohen-Macaulay rings it seems reasonable to simply remove the word "Noetherian" from the definition above. When this is done you must, of course, specify what is meant by the word unmixed (i.e. unmixed or ht-unmixed). You must also specify which type of associated primes you will be using. In this paper we will be restricting our study to the weak Bourbaki. We can then consider the following two definitions as possible appropriate definitions of non-Noetherian Cohen-Macaulay rings.

Definition 2.4. *[1] A ring R is said to be weak Bourbaki unmixed (wB-unmixed) if every height-generated ideal in R is wB-unmixed.*

It can easily be seen that a ring is wB-unmixed if every height-generated ideal I in the ring satisfies $\mathrm{Ass}_f(I) =$Min(I).

Definition 2.5. *[2] A ring R is said to be weak Bourbaki height-unmixed (wB-ht-unmixed) if every height-generated ideal in $R$is wB-ht-unmixed.*

The following lemma follows immediately from lemma 2.1.

Lemma 2.6. *Every LFD wB-ht-unmixed ring is wB-unmixed.*

3 Rings Satisfying PIT or GPIT

One of the most important theorems in Noetherian ring theory is Krull's Principal Ideal Theorem [6]. Krull's theorem says that every Noetherian ring satisfies the following condition (referred to as PIT): If P is a prime ideal minimal over a principal (proper) ideal of R, then ht$(P) \leq 1$.

Theorem 3.1. *Every wB-ht-unmixed (resp. wB-unmixed) domain satisfies PIT.*

Proof. Let R be a domain which does not satisfy PIT. Then $\dim(R) \geq 2$.

Since R does not satisfy PIT, there is a principal ideal $I = (a)$ which has a minimal prime P with $\mathrm{ht}(P) > 1$. Let Q be a prime ideal in R with $\mathrm{ht}(Q) = 1$ and $Q \subset P$ and let x be a non-zero element in Q. Then the product ax is a non-zero element of Q.

Let $J = (ax)$. Then $J \neq (0)$ and $J \subseteq Q$, so $\mathrm{ht}(J) = 1$. Thus, J is height-generated. Consider $(J : x)$. It is easily verified that $(J : x) = (a) = I$. Therefore, since P is minimal over I, we have $P \in \mathrm{Min}(J : x) \subseteq \mathrm{Ass}_f(J)$. Thus, J is not wB-ht-unmixed since $P \in \mathrm{Ass}_f(J)$ with $\mathrm{ht}(P) > 1 = \mathrm{ht}(J)$.

Also, J is not wB-unmixed since $Q \in \mathrm{Min}(J) \subseteq \mathrm{Ass}_f(J)$, so $Q, P \in \mathrm{Ass}_f(J)$, but $Q \subsetneq P$.

Therefore, R is neither wB-ht-unmixed nor wB-unmixed. ∎

As a corollary to theorem 3.1 we get a complete classification of the unmixedness of Prüfer domains. Prüfer domains have been described by Gilmer [7, Chapter IV] as playing a central role in multiplicative ideal theory, so having this characterizations is important to making this unmixedness theory useful.

Corollary 3.2. *A Prüfer domain R is wB-ht-unmixed/wB-unmixed if and only if* $\dim(R) \leq 1$.

Proof. One direction is an immediate consequence of the definition of unmixedness from which it follows that all zero-dimensional rings and all one-dimensional domains are unmixed. (see [1]).

Now, suppose R is an unmixed Prüfer domain. By theorem 3.1, R must satisfy PIT. In [8] it was shown that a Prüfer domain R satisfies PIT if and only if $\dim(R) \leq 1$. Therefore, $\dim(R) \leq 1$. ∎

A more general form of the PIT condition is the GPIT (generalized principal ideal theorem) condition. A ring satisfies GPIT if whenever P is a prime ideal which is minimal over a (proper) ideal generated by n elements, $\mathrm{ht}(P) \leq n$. Every Noetherian ring satisfies GPIT. This fact is often referred to as Krull's generalized principal ideal theorem. One good source for more information on these conditions is the paper by Anderson, Dobbs, Eakin and Heinzer [9]. Our next theorem demonstrates that within the class of rings which satisfy GPIT the concepts of wB-ht-unmixed and wB-unmixed are equivalent.

Theorem 3.3. *If R satisfies GPIT, then R is wB-ht-unmixed if and only if R is wB-unmixed.*

Proof. Suppose R is a ring which satisfies GPIT.

($\Longrightarrow$) Suppose R is wB-ht-unmixed and let I be a height-generated ideal in R.

In a ring which satisfies GPIT every ideal I satisfies $\mathrm{ht}(I) \leq \mu(I)$ where $\mu(I)$ denotes the minimal number of generators of I. Thus, in a ring with GPIT, I is height-generated if and only if $\mathrm{ht}(I) = \mu(I) < \infty$.

To show that I is wB-unmixed, suppose $P, Q \in \mathrm{Ass}_f(I)$ with $P \subseteq Q$. Since R is wB-ht-unmixed we have that I is wB-ht-unmixed. Thus, $\mathrm{ht}(P) = \mathrm{ht}(Q) = \mathrm{ht}(I) < \infty$.

So, P and Q are prime ideals with $P \subseteq Q$ and $\mathrm{ht}(P) = \mathrm{ht}(Q) < \infty$. Thus, $P = Q$ and so I is wB-unmixed.

Therefore, R is wB-unmixed.

($\Longleftarrow$) Suppose R is wB-unmixed and let I be a height-generated ideal in R. As in the first part of this proof, we have $\mathrm{ht}(I) = \mu(I) < \infty$. Let $n = \mathrm{ht}(I) = \mu(I)$. Note that, since R satisfies GPIT, we have $\mathrm{ht}(P) \leq n$ for all $P \in \mathrm{Min}(I)$. However, since $\mathrm{ht}(I) \leq \mathrm{ht}(P)$ for all $P \in \mathrm{Min}(I)$ and $\mathrm{ht}(I) = n$, we have $\mathrm{ht}(P) = n$ for all $P \in \mathrm{Min}(I)$.

Since I is wB-unmixed, we have $\mathrm{Ass}_f(I) = \mathrm{Min}(I)$. Therefore, $\mathrm{ht}(P) = n$ for all $P \in \mathrm{Ass}_f(I)$ and thus, I is wB-ht-unmixed.

Therefore, R is wB-ht-unmixed.

∎

The following result and corollary follow from applying this theorem, and the fact that the condition GPIT localizes, to a result from [1, Theorem 3].

Theorem 3.4. *Let R be a ring with GPIT. If R_M is a wB-ht-unmixed for every maximal ideal M in R, then R is wB-ht-unmixed.*

Corollary 3.5. *If R is a ring such that R_M is a wB-ht-unmixed ring satisfying GPIT (so R_M is also wB-unmixed by theorem 3.3) for every maximal ideal M in R, then R is wB-ht-unmixed (and wB-unmixed).*

Using this corollary we can now show that every locally Cohen-Macaulay ring is both wB-ht-unmixed and wB-unmixed. A ring R is said to be locally Noetherian if R_M is Noetherian for every maximal ideal M in R. Locally Noetherian rings were studied extensively by Heinzer and Ohm [10]. Since GPIT localizes and Noetherian rings satisfy GPIT, we have that every locally Noetherian ring satisfies GPIT.

Example 2.2 from [10] gives an example of a non-Noetherian locally Noetherian domain D whose localizations D_P are all Noetherian valuation domains. Noetherian valuation domains are Cohen-Macaulay, so this is an example of a non-Noetherian ring which is locally Cohen-Macaulay where locally Cohen-Macaulay is defined analogously to locally Noetherian.

Corollary 3.6. *Every locally Cohen-Macaulay ring is wB-ht-unmixed.*

Proof. Suppose R is locally Cohen-Macaulay. Then R_M is Cohen-Macaulay for every maximal ideal M in R.

Cohen-Macaulay rings are Noetherian, therefore each localization R_M is Noetherian and, therefore, satisfies GPIT.

Cohen-Macaulay rings are wB-ht-unmixed (see [2]). so each localization R_M is wB-ht-unmixed.

Apply corollary 3.5 and we have that R is wB-ht-unmixed. ∎

Note that locally Cohen-Macaulay rings are also wB-unmixed (by theorem 3.3) since they satisfy GPIT. Applying this corollary to the example from [10] cited above we see that the example is wB-ht-unmixed. In this case, however, this corollary was unnecessary since the ring D given in the example is a one-dimensional domain and it can easily be shown that, in general, one-dimensional domains are wB-ht-unmixed.

4 Unmixedness of the Ring $k[x_1, x_2, \ldots]$

We would like to have the non-Noetherian ring $k[X_1, X_2, \ldots]$ on our list of non-Noetherian Cohen-Macaulay rings. One reason for this is that, in his work that led to the theory of Cohen-Macaulay rings, Macaulay was studying polynomial rings (in finitely many variables) over a field.

In [1] it was shown that this ring is wB-unmixed by showing that $R[X_1, X_2, \ldots]$ is wB-unmixed for every Cohen-Macaulay domain R. In the proof of this fact (see [1, theorem 4]), the only place where the fact that R is a domain was used was in part (2) of lemma 6. Lemma 4.1 below shows that the assumption that R is a domain was unnecessary. Therefore, we have $R[X_1, X_2, \ldots]$ is wB-unmixed for every Cohen-Macaulay ring R.

Lemma 4.1. *Let R be a Noetherian ring and let $S = R[X_1, X_2, \ldots]$. For any prime ideal P in R we have ht$(P) =$ ht(PS) where ht(PS) refers to the height of the ideal PS in S.*

Note that the proof of this lemma depends only on the weaker condition that R is a strong S-ring (see [6] for more information on strong S-rings). It is not necessary for the ring to be Noetherian.

Proof. First note that we have trivially that $\mathrm{ht}(P) \leq \mathrm{ht}(PS)$ since the extensions of a chain of distinct prime ideals in R is a chain of distinct prime ideals in S.

For $i \geq 1$, let $R_i = R[X_1, X_2, \ldots, X_i]$ so $S = \varinjlim R_i$. Let $P_i = PR_i$. Since R is Noetherian (and thus a strong S-ring) we have $\mathrm{ht}(P) = \mathrm{ht}(P_i)$. [6, theorem 149, page 108]

Now, suppose $\mathrm{ht}(PS) > n$ where $n = \mathrm{ht}(P)$. Then there is a chain of prime ideals

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1} = PS$$

in S. For $1 \leq i \leq n+1$, choose $x_i \in Q_i \setminus Q_{i-1}$. Since $S = \varinjlim R_i$, there is a positive integer j such that $\{x_1, x_2, \ldots, x_{n+1}\} \in R_j$.

For $0 \leq i \leq n+1$, let $T_i = Q_i \cap R_j$. Then

$$T_0 \subset T_1 \subset \cdots \subset T_{n+1}$$

is a chain of prime ideals in R_j. So, $\mathrm{ht}(T_{n+1}) \geq n+1$. However, $T_{n+1} = Q_{n+1} \cap R_j = PS \cap R_j = P_j$ and we have already noted that $\mathrm{ht}(P_j) = n$ so we have a contradiction. Therefore, $\mathrm{ht}(PS) = \mathrm{ht}(P)$. ∎

In [8, Proposition 6.4] it was shown that $R[X_1, X_2, \ldots]$ satisfies GPIT (if R is a Noetherian ring). The statement of this fact in [8] actually makes the assumption that R is a domain, however, the fact that R is a domain, is not necessary in the proof given in [8], so we will use the more general result. By applying theorem 3.3 to this result, we get the following theorem.

Theorem 4.2. *Let R be a Cohen-Macaulay ring. Then $R[X_1, X_2, \ldots]$ is wB-ht-unmixed.*

The desired result regarding the ring $k[X_1, X_2, \ldots]$, where k is a field follows as an easy corollary to this since every field is a Cohen-Macaulay ring.

Corollary 4.3. *For any field k, the ring $k[X_1, X_2, \ldots]$ is a wB-ht-unmixed ring.*

5 Unmixedness and Direct Sums

An interesting result about wB-ht-unmixedness (resp. wB-unmixedness) is that wB-ht-unmixed (resp. wB-unmixed) rings which can be written as a direct sum will have wB-ht-unmixed (resp. wB-unmixed) components.

Theorem 5.1. *Let R be a wB-ht-unmixed (resp. wB-unmixed) ring and assume that R is the direct sum of two rings A and B, so $R = A \oplus B$. Then, both A and B are wB-ht-unmixed (resp. wB-unmixed) rings.*

To prove this theorem we will need some preliminary lemmas.

Lemma 5.2. *Let A and B be commutative rings and let $R = A \oplus B$. Then, for any ideal I in A we have*

$$ht(J) = ht(I)$$

where $J = I \oplus B$.

Lemma 5.2 follows immediately from the fact that any prime ideal in R is of the form $P \oplus B$ for some prime ideal P in A or of the form $A \oplus Q$ for some prime ideal Q in B.

Lemma 5.3. *Let A and B be commutative rings and let $R = A \oplus B$. Then, for any ideal I in A and any $P \in Ass_f(I)$, the prime ideal $Q = P \oplus B$ is in $Ass_f(J)$ where $J = I \oplus B$.*

Proof. Suppose I is an ideal in A and $P \in \mathrm{Ass}_f(I)$. Let $Q = P \oplus B$ and $J = I \oplus B$.

Since $P \in \mathrm{Ass}_f(I)$ there is some element $a \in R$ such that $P \in \mathrm{Min}(I : a)$. We will show that $Q \in \mathrm{Min}(J : (a, 1))$, so $Q \in \mathrm{Ass}_f(J)$.

To show that $Q \in \mathrm{Min}(J : (a, 1))$ we must first verify that $(J : (a, 1)) \subseteq Q$. It is easily verified that $(J : (a, 1)) = (I : a) \oplus B$. So, since $(I : a) \subseteq P$, we have $(J : (a, 1)) = (I : a) \oplus B \subseteq P \oplus B = Q$.

Now, suppose there is a prime ideal T in R such that

$$(J : (a, 1)) \subseteq T \subseteq Q.$$

Then $(I : a) \oplus B \subseteq T \subseteq Q$. Any prime ideal in R is of the form $S \oplus B$ for some prime ideal S in A or of the form $A \oplus S$ for some prime ideal S in B. Since T is a prime ideal and $(I : a) \oplus B \subseteq T$, we have $T = S \oplus B$ for some prime ideal S in A. Note that $(I : a) \subseteq S$. Also, since $S \oplus B = T \subseteq Q = P \oplus B$ we have $S \subseteq P$. Therefore, $S = P$ since P is minimal among prime ideals containing $(I : a)$. Thus, $T = S \oplus B = P \oplus B = Q$, so $Q \in \mathrm{Min}(J : (a, 1))$ and, therefore, $Q \in \mathrm{Ass}_f(I)$. ∎

Now, we are ready to prove theorem 5.1.

Proof. Clearly it is sufficient to prove that if $R = A \oplus B$ is wB-ht-unmixed (resp. wB-unmixed), then A is wB-ht-unmixed (resp. wB-unmixed).

Let I be a height-generated ideal in A. Then $I = (a_1, a_2, \ldots, a_n)$ for some $a_1, a_2, \ldots, a_n \in A$ where $\mathrm{ht}(I) \geq n$.

Let $J = I \oplus B$. Then J is generated in R by the elements $(a_1, 1), (a_2, 1), \ldots, (a_n, 1)$. Also, by lemma 5.2, $\mathrm{ht}(J) = \mathrm{ht}(I)$. Therefore, J is height-generated.

Let $P, Q \in \mathrm{Ass}_f(I)$ with $P \subseteq Q$. Then, by lemma 5.3, $P \oplus B, Q \oplus B \in \mathrm{Ass}_f(J)$. (Also, of course, $P \oplus B \subseteq Q \oplus B$.)

- If we are assuming that R is wB-ht-unmixed, then, since J is height-generated in R and since $P \oplus B \in \mathrm{Ass}_f(J)$, we have $\mathrm{ht}(P \oplus B) = \mathrm{ht}(J)$. By lemma 5.2, $\mathrm{ht}(P) = \mathrm{ht}(P \oplus B)$. Therefore, $\mathrm{ht}(P) = \mathrm{ht}(P \oplus B) = \mathrm{ht}(J) = \mathrm{ht}(I)$. So, A is wB-ht-unmixed.
- On the other hand, if we are assuming that R is wB-unmixed, then, since J is height-generated in R and $P \oplus B, Q \oplus B \in \mathrm{Ass}_f(J)$ with $P \oplus B \subseteq Q \oplus B$, we have $P \oplus B = Q \oplus B$. Therefore, $P = Q$. So, A is wB-unmixed.

∎

The converse of this theorem (theorem 5.1) is true if the ring R is assumed to satisfy GPIT. It is easily seen that for $R = A \oplus B$, we have R satisfies GPIT if and only if A and B satisfy GPIT. Thus, it is sufficient to assume that R satisfies GPIT or that A and B satisfy GPIT to get the converse. Also, since R, A and B all satisfy GPIT in this theorem, wB-ht-unmixed is equivalent to wB-unmixed, so we will not have to consider two cases as we did in the proof of theorem 5.1 above.

Theorem 5.4. *Suppose R is the direct sum of two rings A and B, so $R = A \oplus B$. Assume R satisfies GPIT (or equivalently that both A and B satisfy GPIT). Then R is wB-ht-unmixed (resp. wB-unmixed) if and only if A and B are both wB-ht-unmixed (resp. wB-unmixed).*

Proof. One direction of this theorem follows immediately from theorem 5.1. Also, as was mentioned above, it is sufficient to prove the wB-ht-unmixed version of this theorem. We need to show that if A and B are wB-ht-unmixed rings, then $R = A \oplus B$ is wB-ht-unmixed.

Suppose A and B are wB-ht-unmixed and that $R = A \oplus B$ satisfies GPIT (so A and B also satisfy GPIT).

Throughout the following we will make use of the following notation. If J is an ideal in R, we define $J_A = \{a \in A : (a,b) \in J \text{ for some } b \in B\}$ and $J_B = \{b \in B : (a,b) \in J \text{ for some } a \in A\}$. It is easy to see that $J \subseteq J_A \oplus J_B$ and, in fact, $J \subseteq S \oplus T$ (where S is an ideal in A and T is an ideal in B) if and only if $J_A \subseteq S$ and $J_B \subseteq T$.

Let I be a height-generated ideal in R of height n. Then

$$I = ((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$$

for some $a_1, a_2, \ldots, a_n \in A$ and some $b_1, b_2, \ldots, b_n \in B$. It follows easily that $I_A = (a_1, a_2, \ldots, a_n)$ and $I_B = (b_1, b_2, \ldots, b_n)$.

Recall that every prime ideal in R is of the form $P \oplus B$ or $A \oplus Q$ where $P \in \operatorname{Spec}(A)$ and $Q \in \operatorname{Spec}(B)$, so

$$\begin{aligned} n &= \operatorname{ht}(I) = \min\{\operatorname{ht}(T) : T \in \operatorname{Spec}(R) \text{ and } I \subseteq T\} \\ &= \min \{\min\{\operatorname{ht}(P \oplus B) : P \in \operatorname{Spec}(A) \text{ and } I \subseteq P \oplus B\}, \\ &\qquad \min\{\operatorname{ht}(A \oplus Q) : Q \in \operatorname{Spec}(B) \text{ and } I \subseteq A \oplus Q\}\} \end{aligned}$$

It is easily seen (see lemma 5.2) that, for $P \in \operatorname{Spec}(A)$ and $Q \in \operatorname{Spec}(B)$, we have $\operatorname{ht}(P \oplus B) = \operatorname{ht}(P)$ and $\operatorname{ht}(A \oplus Q) = \operatorname{ht}(Q)$. Also, $I \subseteq P \oplus B$ if and only if $I_A \subseteq P$ and $I \subseteq A \oplus Q$ if and only if $I_B \subseteq Q$. Therefore,

$$\begin{aligned} n &= \min \{\min\{\operatorname{ht}(P) : P \in \operatorname{Spec}(A) \text{ and } I_A \subseteq P\}, \\ &\qquad \min\{\operatorname{ht}(Q) : Q \in \operatorname{Spec}(B) \text{ and } I_B \subseteq Q\}\} \\ &= \min\{\operatorname{ht}(I_A), \operatorname{ht}(I_B)\}. \end{aligned}$$

This gives us that $n \leq \operatorname{ht}(I_A)$ and $n \leq \operatorname{ht}(I_B)$. However, I_A and I_B are both ideals which can be generated by n elements in the GPIT satisfying rings A and B respectively. So, $\operatorname{ht}(I_A) \leq n$ and $\operatorname{ht}(I_B) \leq n$. Therefore, I_A and I_B are height-generated ideals in their respective rings of height n. Thus, since A and B are both assumed to be wB-ht-unmixed rings we have $\operatorname{ht}(P) = n$ for all $P \in \operatorname{Ass}_f(I_A)$ and $\operatorname{ht}(Q) = n$ for all $Q \in \operatorname{Ass}_f(I_B)$.

Suppose $T \in \operatorname{Ass}_f(I)$. We need to show $\operatorname{ht}(T) = n$.

Since T is a prime ideal $T = P \oplus B$ for some $P \in \operatorname{Spec}(A)$ or $T = A \oplus Q$ for some $Q \in \operatorname{Spec}(B)$. Without loss of generality, suppose $T = P \oplus B$ for some $P \in \operatorname{Spec}(A)$. Since $T \in \operatorname{Ass}_f(I)$, we have $T \in \operatorname{Min}(I : (a,b))$ for some $(a,b) \in R$. So, $P \oplus B \in \operatorname{Min}(I : (a,b))$. This will be true if and only if $P \in \operatorname{Min}(I : (a,b))_A$.

Claim: $(I : (a,b))_A = (I_A : a)$

- Note that

$$\begin{aligned} (I : (a,b)) &= \{(x,y) : (ax, by) \in I\} \\ &\subseteq \{(x,y) : ax \in I_A \text{ and } by \in I_B\} \\ &= \{(x,y) : x \in (I_A : a) \text{ and } y \in (I_B : b)\} \\ &= (I_A : a) \oplus (I_B : b) \end{aligned}$$

  Therefore, $(I : (a,b))_A \subseteq (I_A : a)$.

- Also note that

$$\begin{aligned} x \in (I_A : a) &\implies ax \in I_A \\ &\implies (ax, y) \in I \text{ for some } y \in B \\ &\implies (ax, y)(1, b) \in I \text{ for some } y \in B \\ &\implies (ax, by) \in I \text{ for some } y \in B \\ &\implies (x, y) \in (I : (a,b)) \text{ for some } y \in B \\ &\implies x \in (I : (a,b))_A \end{aligned}$$

Therefore $(I_A : a) \subseteq (I : (a,b))_A$.

Thus, the claim has been proven.

So, $P \in \mathrm{Min}(I_A : a)$ and therefore, $P \in \mathrm{Ass}_f\,(I_A)$. Thus, $\mathrm{ht}(P) = \mathrm{ht}(I_A) = n$. Which give us $\mathrm{ht}(T) = \mathrm{ht}(P) = n = \mathrm{ht}(I)$. Thus, I is wB-ht-unmixed.

Therefore, R is wB-ht-unmixed. ∎

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