Deduction
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Chapter 4 Natural Deduction

Natural deduction and proof

- A natural deduction system is a set of inference rules used to derive sentences from sets of other sentences.
  - Hypothetical proofs derive sentences from sets containing one or more assumptions.
  - Categorical proofs derive sentences from sets containing no assumptions. (In set theory, a set containing no members is called the null set.)

Rudiments of proof

- Every proof begins with a set of assumptions followed by the show line. (If it is a categorical proof, then it simply starts with a show line.)
- All successive lines of the proof are generated and explicitly justified by the application of inference rules (which we will learn).
- When a proof arrives at the sentence contained in the show line, the show line is cancelled. A cancelled show line means that the proof of the sentence to be shown is finished.

Direct proof

- We will be learning a few different types of proof. The simplest is called a direct proof. A direct proof with three assumptions would be set up like this.

  1. $\mathcal{A}$ A
  2. $\mathcal{B}$ A
  3. $\mathcal{C}$ A
  4. Show $\mathcal{D}$

- The A's on the right hand side of the assumptions just indicate that the sentences are assumptions.
- Our first rule is actually the assumption rule (p.110). It tells you that you can take on any sentence at all as an assumption prior to the show line, and that it must be identified as such as above.

Finished direct proofs

- A finished direct proof will look like this.

1.	$\mathcal{A}$	A
2.	$\mathcal{B}$	A
3.	$\mathcal{C}$	A
4.	~~Show~~ $\mathcal{D}$	
.		Rule
.		Rule
.		Rule
.		Rule
n .	$\mathcal{D}$	Rule

- The lines between the show line and the final line of the proof are all numbered consecutively. Like the assumption lines, each consecutive line is accompanied by a rule that justifies it on the right. The entire derivation is bracketed by a vertical line.

Rules of inference

- Like truth tables for connectives and truth tree rules, inference rules in a natural deduction system just have to be memorized.
- Fortunately, most of them are fairly intuitive. All of them can be checked for their validity using truth tables.

Rules for &

- **There are two inference rules for &**
  - **Conjunction Exploitation** (&E)

$$\frac{\mathcal{A} \,\&\, \mathcal{B}}{\mathcal{A} \text{ (or } \mathcal{B})}$$

  - **Summary**: This rule just says that if you have a formula of the form ($\mathcal{A}$ & $\mathcal{B}$) you can write either one of the conjuncts $\mathcal{A}$ or $\mathcal{B}$, on a line by itself.
  - **Conjunction Introduction** (&I)

$$\begin{array}{l} \mathcal{A} \\ \underline{\mathcal{B} \qquad\quad} \\ \mathcal{A} \,\&\, \mathcal{B} \end{array}$$

  - **Summary**: This rule says that if you have two formulas on two separate lines of the proof, then you can join them in a conjunction on a separate line.

Negation Introduction/Exploitation ¬ ¬

- There are two rules for ¬. The first is:
  - **Negation Introduction/Exploitation** (¬¬)

$$\frac{\neg\neg\mathcal{A}}{\mathcal{A}}$$

  - **Summary**: This rule says that if you have a formula of the form $\neg\neg\mathcal{A}$ on a line by itself you can write $\mathcal{A}$ on a line by itself. It also allows you to go the other direction; i.e., If you have $\mathcal{A}$ on a line by itself you can write $\neg\neg\mathcal{A}$.
  - **Note**: The double line between the formulas means the rule in invertible: it can go both ways. If there is just a single line then it only goes top to bottom.

Our first proof

- Let's do a proof using these rules

1.	(p & q)	A
2.	¬¬ r	A
3.	Show: (q & r)	

Our first proof 2

1. (p & q) A
2. ¬¬ r A
3. Show: (q & r)
4. q &E, 1

Note: This says that line 4 is justified by the rule of &E performed on line 1. You must always indicate which line of the proof the rule is being applied to.

Our first proof 3

1.	(p & q)	A
2.	¬¬ r	A
3.	Show: (q & r)	
4.	q	&E, 1
5.	r	¬¬, 2

Note: This says that line 5 is justified by the rule of ¬¬ performed on line 2.

Our first proof 4

1.	(p & q)	A
2.	¬¬ r	A
3.	Show: (q & r)	
4.	q	&E, 1
5.	r	¬¬, 2
6.	(q & r)	&I, 4,5

Note: This says that line 6 is justified by the rule of &I performed on lines 4 & 5.

Our first proof 5

1.	(p & q)	A
2.	¬¬ r	A
3.	~~Show~~: (q & r)	
4.	q	&E, 1
5.	r	¬¬, 2
6.	(q & r)	&I, 4,5

Note: We now have the desired result. So we indicate that by bracketing the derivation and canceling the show line.

Indirect Proof

- The second rule involving negation is also a distinct method of proof known as <u>indirect proof</u>. Classically, indirect proof is known by the Latin phrase: *reductio ad absurdum*, or just *reductio* for short.
- Before stating the rule, we should state the idea behind indirect proof. The idea is the same as the one behind the truth tree method, viz., that if an argument is valid, denying the conclusion will result in a contradictory set of sentences.
- Mechanically, an indirect proof works like this. If a show line has a negated formula on it, then assume the non negated formula on the next line. Then try to derive a contradiction. When a formula and it’s negation occur on separate lines of the proof, you can cancel the original show line.

Indirect proof rule

- The rule for indirect proof is executed as follows. (p. 110).

Show $\neg \mathcal{A}$

$\mathcal{A}$ AIP

.

.

.

$\mathcal{B}$

$\neg \mathcal{B}$

- **Note**: AIP stands for “assumption for indirect proof.” The formula $\mathcal{B}$ and it’s negation $\neg \mathcal{B}$ must occur on successive lines of the proof. Also, remember that the formula $\mathcal{B}$ is not necessarily atomic. Absolutely any contradiction will satisfy the indirect proof rule. For example, $\mathcal{B}$ could be $(p \rightarrow q)$, in which case $\neg \mathcal{B}$ would simply be $\neg(p \rightarrow q)$.

Example of indirect proof

- Let’s prove the following indirectly

1. $p \,\&\, \neg p$ A
2. Show $\neg q$

Example of indirect proof 2

1. $p \mathbin{\&} \neg p$ A
2. Show $\neg q$
3. q AIP

Note: This is just the first step of indirect proof, i.e., remove the ¬ from the negated conclusion and assume the resulting formula.

Example of indirect proof 3

1.	p & ¬ p	A
2.	Show ¬ q	
3.	q	AIP
4.	p	&E, 1
5.	¬p	&E, 1

Note: Steps 4 and 5 are just sequential applications of &E on line 1.

Example of indirect proof 4

1.	p & ¬ p	A
2.	~~Show~~ ¬ q	
3.	q	AIP
4.	p	&E, 1
5.	¬p	&E, 1

Note: You now have a contradiction on lines 4 and 5, so you're done. This proof strikes most people as pretty weird. It shows what's bad about contradictions. Namely, that they allow you to derive anything at all.

The rule of reiteration

- Reiteration is a rule that says anytime you have a formula on a line, you can write that formula again on a different line anytime you want. It's mainly useful for complying with the requirement of indirect proof to locate the contradiction in the last two lines of the proof.
- Reiteration

$$\frac{\mathcal{A}}{\mathcal{A}}$$

Conditional exploitation $\to$E

- There are two rules for the conditional. The first one is traditionally known as *modus ponens*, though we'll call it conditional exploitation.

$$\begin{array}{l} \mathcal{A} \to \mathcal{B} \\ \underline{\mathcal{A} \qquad\quad} \\ \mathcal{B} \end{array}$$

- Whenever you employ $\to$E you must cite both the line on which $\mathcal{A} \to \mathcal{B}$ occurs and the line on which $\mathcal{A}$ occurs.

Conditional proof

- Conditional proof is our third and final method of proof. It is used only in attempting to derive a sentence whose main connective is a conditional.
- The basic idea of conditional proof is simple. If you want to prove a statement of the form $\mathcal{A} \rightarrow \mathcal{B}$ you may simply assume $\mathcal{A}$ and then try to derive $\mathcal{B}$.
- So, like indirect proof, you make a temporary assumption, which is eliminated when the derivation is bracketed off and the show line is cancelled.

Conditional proof rule

The rule for conditional proof works like this

~~Show~~ $\mathcal{A} \to \mathcal{B}$

| $\mathcal{A}$ ACP

| .

| .

| .

| $\mathcal{B}$

ACP means “assume for conditional proof.”

Example of conditional proof and →E

- Here is an example of a proof that uses both →E and conditional proof.

1. $p \rightarrow q$	A
2. $p \rightarrow r$	A
3. Show $p \rightarrow (q \;\&\; r)$	
4. p	ACP

Example of conditional proof and $\to$E

1. $p \to q$	A
2. $p \to r$	A
3. Show $p \to (q \ \& \ r)$	
4. p	ACP
5. q	$\to$E, 1,4
6. r	$\to$E, 2,4

Example of conditional proof and →E

1. $p \rightarrow q$	A
2. $p \rightarrow r$	A
3. Show $p \rightarrow (q \& r)$	
4. p	ACP
5. q	→E, 1,4
6. r	→E, 2,4
7. $q \& r$	&I, 5,6

Example of conditional proof and →E

1.	$p \rightarrow q$	A
2.	$p \rightarrow r$	A
3.	~~Show~~ $p \rightarrow (q \;\&\; r)$	
4.	p	ACP
5.	q	→E, 1,4
6.	r	→E, 2,4
7.	$(q \;\&\; r)$	&I, 5,6

Be sure to study the structure of this proof. You are able to cancel the show line because you proved the conditional, first by assuming p on line 4 and by deriving (q&r) on line 7.

Biconditional introduction $\leftrightarrow$ I

- There are two simple rules for the biconditional. Biconditional introduction $\leftrightarrow$ I is analogous to &I and works like this.

$$\begin{array}{c} \mathcal{A} \rightarrow \mathcal{B} \\ \underline{\mathcal{B} \rightarrow \mathcal{A}} \\ \mathcal{A} \leftrightarrow \mathrm{B} \end{array}$$

Biconditional exploitation ↔ E

- Biconditional exploitation ↔ E is analogous to → E and it works like this.

$$\mathcal{A} \leftrightarrow \mathcal{B}$$

$$\underline{\mathcal{A} \text{ (or } \mathcal{B}\text{)}}$$

$$\mathcal{B} \text{ (or } \mathcal{A}\text{)}$$

A proof using $\leftrightarrow$ E

- We'll now do a proof using biconditional exploitation and <u>multiple</u> show lines.

1. $p \leftrightarrow q$ A
2. Show $(p \rightarrow q) \& (q \rightarrow p)$

In order to do this proof we will prove each conjunct separately. So we'll begin by writing another show line.

A proof using $\leftrightarrow$ E

1. $p \leftrightarrow q$ A

2. Show $(p \rightarrow q)$ & $(q \rightarrow p)$

3. Show $(p \rightarrow q)$

We will now proceed to demonstrate $(p \rightarrow q)$ by assuming p for conditional proof, and deriving q by $\leftrightarrow$ E.

A proof using $\leftrightarrow$ E

1. $p \leftrightarrow q$ A
2. Show $(p \rightarrow q) \& (q \rightarrow p)$
3. Show $(p \rightarrow q)$
4. p ACP
5. q $\leftrightarrow$ E 1,4

We have now demonstrated $(p \rightarrow q)$. So we bracket this portion of the proof, cancel the show on line 3, write a new show line on line 6, and repeat this process for $(q \rightarrow p)$.

A proof using ↔ E

1.	$p \leftrightarrow q$	A
2.	Show $(p \rightarrow q)$ & $(q \rightarrow p)$	
3.	~~Show~~ $(p \rightarrow q)$	
4.	p	ACP
5.	q	$\leftrightarrow$ E 1,4
6.	~~Show~~ $(q \rightarrow p)$	
7.	q	ACP
8.	p	$\leftrightarrow$ E 1,7

Because the show lines on 3 and 6 are now canceled, we are entitled to the corresponding conditionals. The last step, then is to simply join them in a conjunction using &I.

A proof using ↔ E

1.	$p \leftrightarrow q$	A
2.	~~Show~~ $(p \rightarrow q) \& (q \rightarrow p)$	
3.	~~Show~~ $(p \rightarrow q)$	
4.	p	ACP
5.	q	$\leftrightarrow$ E 1,4
6.	~~Show~~ $(q \rightarrow p)$	
7.	q	ACP
8.	p	$\leftrightarrow$ E 1,7
9.	$(p \rightarrow q) \& (q \rightarrow p)$	&I, 3,6

This is the conclusion we were looking for, so we are now entitled to bracket off the entire proof and cancel the original show line, which finishes the proof.

Caveats on the use of show lines

- Here is one very important thing to understand about proofs with multiple show lines:
  - Once a show line has been canceled, the lines that have been bracketed off beneath it <u>are dead</u> and may not be used for any other purpose in the proof.
  - Also, anytime you add a <u>new</u> show line, your project now must become narrowly focused on deriving whatever is written on that show line, not what is written in the original show line.

Caveats on the use of show lines.

1.	$p \leftrightarrow q$	A
2.	~~Show~~ $(p \rightarrow q) \& (q \rightarrow p)$	
3.	~~Show~~ $(p \rightarrow q)$	
4.	p	ACP
5.	q	$\leftrightarrow$ E 1,4
6.	~~Show~~ $(q \rightarrow p)$	
7.	q	ACP
8.	p	$\leftrightarrow$ E 1,4
9.	$(p \rightarrow q) \& (q \rightarrow p)$	&I, 3,6

For example, once you have bracketed lines 4-5 and canceled line 3, you may only use what has been shown on line 3 below. You may no longer use lines 4 or 5.

Caveats on the use of show lines.

1.	$p \leftrightarrow q$	A
2.	~~Show~~ $(p \rightarrow q) \& (q \rightarrow p)$	
3.	~~Show~~ $(p \rightarrow q)$	
4.	p	ACP
5.	q	$\leftrightarrow$ E 1,4
6.	~~Show~~ $(q \rightarrow p)$	
7.	q	ACP
8.	p	$\leftrightarrow$ E 1,7
9.	$(p \rightarrow q) \& (q \rightarrow p)$	&I, 3,6

Also, after writing the new show line on 3, your new sub project is to derive $(p \rightarrow q)$, not the original show line on 2.

The same point holds true of the show line on 6.

Two more caveats

- There are two more basic points worth revisiting right now.

1. We never <u>prove</u> our <u>assumptions</u>. Assumptions are what we <u>use</u> to prove other conclusions.
2. The formula on a *Show* line is what we are <u>trying to</u> <u>prove</u>. Hence, the formula on the show line is not itself an assumption to be used for proving anything else unless the *Show* line has been cancelled.

A bogus proof

- Based on what we just said, what is wrong with this proof?

1. $p \leftrightarrow q$ A
2. ~~Show~~ $(p \rightarrow q) \& (q \rightarrow p)$
3. | $(p \rightarrow q)$ &E, 2
4. | $(q \rightarrow p)$ &E, 2
5. | $(p \rightarrow q) \& (q \rightarrow p)$ &I, 3,4

Another bogus proof

What's wrong with this proof?

1.	$p \rightarrow q$	A
2.	$r \rightarrow s$	A
3.	~~Show~~ $(p \& r) \rightarrow (q \& s)$	
4.	~~Show~~ $(q \& s) \rightarrow (q \& s)$	
5.	$(q \& s)$	ACP
6.	q	&E, 5
7.	s	&E, 5
8.	$(q \& s)$	

Previous proof done proper

1.	$(p \rightarrow q)$	A
2.	$(r \rightarrow s)$	A
3.	~~Show~~ $(p \,\&\, r) \rightarrow (q \,\&\, s)$	
4.	$(p \,\&\, r)$	ACP
5.	p	&E, 4
6.	r	&E, 4
7.	q	$\rightarrow$E, 1,5
8.	s	$\rightarrow$E, 2,6
9.	$q \,\&\, s$	&I, 7,8

Disjunction introduction ∨I

- There are two rules for disjunction. The first rule, ∨I, works like this

$$\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}}$$

This will strike you as a strange rule until you recall the truth table for disjunction. So just remember that a disjunction is true anytime just one of the disjuncts is true. That means that if we know that $\mathcal{A}$ is true, we know that $\mathcal{A}$ or <u>anything else at all</u> is true as well.

Disjunction exploitation vE

- Disjunction exploitation vE is the most complicated rule of all, but it isn't too hard to understand.
- The basic idea is that in order to prove that something follows from a disjunction you have to show that it follows from <u>each</u> of the disjuncts separately.
- In other words, if you want to show that some formula C, follows from the disjunction $\mathcal{A}$ v $\mathcal{B}$, then you need to show $\mathcal{A} \rightarrow C$ and you also need to show that $\mathcal{B} \rightarrow C$.
- The reason for this, again, is the truth conditions of $\mathcal{A}$v$\mathcal{B}$. Sentences of this form are true if only <u>one</u> of the disjuncts is true. So if all you know is that one of the disjuncts is true, then in order to be able to derive anything solid from the disjunction you have to show that if follows from <u>both</u> disjuncts independently.

The rule of vE

- The rule for vE is:

$$\mathcal{A} \vee \mathcal{B}$$
$$\mathcal{A} \rightarrow \mathcal{C}$$
$$\underline{\mathcal{B} \rightarrow \mathcal{C}}$$
$$\mathrm{C}$$

- There are basically two things that make this rule a little complicated to use.
  - First, there are 3 lines involved in its justification.
  - Second, the two conditionals often themselves have to be proven independently.

An example using vI and vE

- Here is an example using both of our new disjunction rules.

1.	$(p \to q)$	A
2.	$(\neg r \to s)$	A
3.	$(s \to q)$	A
4.	$(p \vee \neg r)$	A
5.	Show $(q \vee t)$	

- The basic strategy of this proof is to observe that it is possible to derive q from the first four assumptions using vE. Once q is established, you may simply add t using vI. (Of course, t doesn't occur anywhere in the assumptions, but vI doesn't require that.)

An example using vI and vE

1. $(p \rightarrow q)$ A
2. $(\neg r \rightarrow s)$ A
3. $(s \rightarrow q)$ A
4. $(p \vee \neg r)$ A
5. Show $(q \vee t)$
6. Show $(\neg r \rightarrow q)$

- You can derive line 6 using conditional proof and successive applications of →E on lines 2 and 3.
- The reasons you <u>want</u> to show line 6 is that it, in combination with line 1 and 4 will allow you to use vE to demonstrate that q.

An example using vI and vE

1.	$(p \rightarrow q)$	A
2.	$(\neg r \rightarrow s)$	A
3.	$(s \rightarrow q)$	A
4.	$(p \lor \neg r)$	A
5.	Show $(q \lor t)$	
6.	~~Show~~ $(\neg r \rightarrow q)$	
7.	$\neg r$	ACP
8.	s	$\rightarrow$E, 2,7
9.	q	$\rightarrow$E, 3,8

You have now have everything you need to us the rule of VE. Lines 4, 1 & 6 jointly imply q by vE.

An example using vI and vE

1.	$(p \rightarrow q)$	A
2.	$(\neg r \rightarrow s)$	A
3.	$(s \rightarrow q)$	A
4.	$(p \vee \neg r)$	A
5.	Show $(q \vee t)$	
6.	~~Show~~ $(\neg r \rightarrow q)$	
7.	$\neg r$	ACP
8.	s	$\rightarrow$E, 2,7
9.	q	$\rightarrow$E, 3,8
10.	q	vE, 4, 1, 6

If you find step 10 confusing here, it's probably because you don't quite understand the rule of vE yet. (Go back and review it.) The other thing that looks a little weird is that q occurs on <u>both</u> lines 9 and 10. Is that redundant? No. On line 9, q has only been derived to establish the conditional on line 6. You have not shown q <u>itself</u> until you show that it follows by vE from lines 4,1, & 6.

An example using vI and vE

1.	$(p \to q)$	A
2.	$(\neg r \to s)$	A
3.	$(s \to q)$	A
4.	$(p \lor \neg r)$	A
5.	~~Show~~ $(q \lor t)$	
6.	~~Show~~ $(\neg r \to q)$	
7.	$\neg r$	ACP
8.	s	$\to$E, 2,7
9.	q	$\to$E, 3,8
10.	q	vE, 4, 1, 6
11.	$q \lor t$	vI, 10

The final step of this proof is to simply add t by vI.

That’s it!

- That is all of the basic rules of sentential logic. This set of rules forms a <u>complete</u> system that allows you to prove absolutely every argument form that is valid in sentential logic.
- It also is sound, which means that it if you use it correctly you’ll never be able to ‘prove’ an invalid argument form valid.
- However, when all you have is the basic rules, proving some argument forms to be valid is extremely difficult. So it is going to help to know some other derivable, and more powerful rules as well. We’ll do that next.