Determination of Odds in Prediction Markets: Coexistence of Posted-offer and Double-auction Designs*

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Abstract

This paper offers a theoretical model of coexistence of two competing mechanisms in the same market; one follows the posted-offer rule and the other one incorporates a double-auction mechanism. We motivate the study of this coexistence with a sports betting example; bettors are free to choose which mechanism they want to place their bets in. The model implies that *i*) bettors' risk aversion parameter is instrumental in whether these two mechanisms coexist or not, *ii*) most bettors are strictly better off, and none is worse off, when they have access to both of these competing mechanisms rather than just one, and *iii*) these results hold even when we allow the posted-offer market maker (bookie) to make a positive profit instead of following a zero expected profit pricing rule.

Keywords: Prediction markets, posted-offer, double-auction.
JEL Classification: D44, D82.

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1 Introduction

> "Whoever being engaged in the business of betting or wagering knowingly uses a wire communication facility for the transmission in interstate or foreign commerce of bets or wagers or information assisting in the placing of bets or wagers on any sporting event or contest, or for the transmission of a wire communication which entitles the recipient to receive money or credit as a result of bets or wagers, or for information assisting in the placing of bets or wagers, shall be fined under this title or imprisoned not more than two years, or both." Interstate Wire Act (1961)

The 1961 Interstate Wire Act, commonly known as the *1961 Wire Act*, has made it clear that bettors in the United States can not use credit cards or money transfer companies to transmit funds to their accounts with online sports betting websites. While many online sportsbooks have found a way around this law (through basing the company outside the U.S. and staying there), many big credit card companies decline to complete transactions which indicate that funds might be used for online sports betting.[1] This paper is partly motivated by what is a direct consequence of this law, that without online sportsbooks in the markets, the only access bettors have in the U.S. are traditional bookies. As a result, bettors miss out on access to another type of betting mechanism, explained below, that is deep, liquid and is shown to offer better odds. We not only show that both types of mechanisms could coexist in the same market (if legal issues were not an obstacle), but also that bettors who have access to both are in average better off than those whose only avenue for betting is through bookies.[2]

Whether it is sports betting, financial markets or retail industry, trading mechanisms in markets can mainly be categorized into two main types in term of their structure. The main type of mechanism in the retail industry is known to follow the posted-offer rule; the seller posts prices and the buyer makes a decision of whether to buy or not. Another type, which is more prevalent among financial markets, is known as double auction (DA) mechanism. In DA setup, buyers and sellers post prices simultaneously and whenever there is a match, trade occurs. The versatility of DA setup is that at any point during trading period (and some markets such as foreign exchange markets are operational around the clock), buyers and sellers can observe the highest buy and sell price, post new offers which are queued and could be accepted when another bettor who is willing to take its counterpart is found, and accept outstanding buy or sell offers by taking the other end of the deal.

In the literature, experiments have been the main method of comparison for these two trading mechanisms in markets.[3] Many experimental studies conclude that convergence to the competitive price is quite rapid under DA, whereas in posted-offer the price either converges more slowly or does not converge at all. Even though findings of these experimental studies have been quite exciting, literature has been lacking a robust theoretical model to accompany these findings. Our contribution is to introduce the theoretical model that has been missing in the literature so far. To the best of our knowledge, no attempt has been made to model the coexistence of posted-offer and DA mechanisms competing in the same market.

The reasoning in our model is as follows. By design, our model allows bettors the freedom to choose between either mechanism. In our theoretical model, after bettors observe the prices quoted by the bookie, they have to make their choice. They compare the utility they will get from betting with the bookie with the utility they expect to get from placing their bet with the betting exchange. Because there is price uncertainty in the betting exchange through the noise we introduce in the model, bettors use their expectations about what the price will be in the betting exchange; they do not have exact knowledge of it at the time they make their choice. Unfortunately, this model does not have a closed-form solution but we use numerical solution techniques to find bettors' optimal decisions. We show that bettors who are moderately risk averse with respect to their wealth always choose to place their bets with the betting exchange. When bettors are assumed to be more risk averse, especially above a certain threshold, we find that bettors with extreme beliefs about the outcome of the bet choose to bet with the bookie; and all other bettors choose to go to the betting exchange to place their bets. These results hold even when we relax bookie's zero-profit assumption and allow him to quote odds that will get him sure positive profits.

Since betting exchanges are examples of the DA format applied to sports wagering markets, let us motivate by continuing to use the sports betting setup with a bookie and a betting exchange.[4] When placing a bet with a bookie, bettors would have to accept whatever is offered to them; a bookie would post odds and bettors, upon observing these odds, would decide to wager or not, and by how much. On the contrary, bettors post their odds and amounts they wish to trade in a betting exchange. As long as another bettor takes the opposite end of a posted deal, that bet is matched. A small house fee can be charged by both establishments but generally DA markets charge a relatively lower fee than bookies do.

A quick look at the leading sites' odds seems to also reveal that betting exchanges offer better prices, even after the 3% commission is taken into account.[5] As of June 30, 2010, on the largest betting exchange, Betfair, a bet on Brazil as the 2010 World Cup winner pays 3.5 to 1, Spain pays 4 to 1, Argentina pays 6 to 1, and Germany pays 8.6 to 1. At a leading bookmaker, William Hill, the odds are 3.5, 3.75, 5.5 and 7.5, respectively. The returns on these four bets are higher by about 4.3% at Betfair. If we look at the odds for the top goal scorer in that same competition, at Betfair, David Villa pays 2.98 to 1, Gonzalo Higuain pays 4.3 to 1, Luis Fabiano pays 6.8 to 1, and Luis Suarez pays 16 to 1. The odds at William Hill are 2.75, 3.5, 5 and 15, respectively. The returns on these four bets are higher by about 18.5% at Betfair. Note that since predicting 2010 World Cup winner is a much more popular bet than the bet on the top goal scorer, price improvement observed in the betting exchange is not very high. A similar observation is also found in Ozgit (2006).

If we assume that betting exchanges offer prices that are at least as good as prices offered by bookies, then why is there still trading with bookies? Moreover, bookies still have a considerable size of the market as well. There are some explanations for this observation in the literature, namely that *i*) betting exchange markets are thin, so the big bettor has to go to bookies, *ii*) betting exchanges are new and some proportion of bettors are unaware of them (or they cannot have access to them due to legal matters), *iii*) because bookies were active businesses long before betting exchanges were, bettors might have brand loyalty. While these explanations are all valuable points, our model proposes a fourth explanation born out of economic decision making; that is some bettors choose to place their bets with the bookie as a result of their expected utility maximization.

The remainder of the paper proceeds as follows. We present an extensive literature review in section 2. Section 3 describes the setup of the model, presents the decision mechanism of bettors and outlines how equilibrium is reached in both betting markets. Section 4 demonstrates how equilibrium is reached under different parameter values and shows that both betting markets can coexist. Section 5 concludes and offers a policy recommendation based on the results.

2 Literature Review

Our paper can be categorized into two strands of literature; sports betting and coexistence of DA and posted-offer mechanisms in the same market. In terms of the topic of sports betting, our paper asks a different question than most existing studies in prediction and betting markets do. Unlike many existing studies, we do not try to explain the longshot bias in betting markets and we also do not try to analyze how well prediction markets predict the actual outcome of an event. Our main objective is to show how a bettor can be better off when he has access to a bookie as well as a betting exchange. In terms of the coexistence of DA and posted-offer markets, our model is analogous to a few papers that have studied similar setups in the past but unlike them, we introduce risk aversion and show how instrumental bettors' level of risk aversion is for the coexistence of these two mechanisms.

In terms of the first strand of literature mentioned above, Smith et. al. (2006) is closest to our research question. They study the efficiency gains brought on by betting exchanges using data from UK horse racing. They compare matching data from betting exchanges as well as traditional bookie markets related to 799 horse races ran in the UK during 2002. They find that betting exchanges are found to have significantly lower market bias compared to traditional bookie markets. This means betting exchanges supply efficiency gains by lowering transaction costs for consumers.

Ozgit (2006) makes use of a unique dataset of bets placed with a betting exchange, as well as with bookies, on National Basketball Association games played between December 2004 and February 2005. He finds that the betting exchange in his study, Betfair, offers significantly better prices than what bookies offer for small bets. As bet size increases, Ozgit finds that difference in prices offered drops significantly. He also finds that bettors are likely to use bookies when liquidity on the betting exchange is low, hence identifying the importance of liquidity in bettors' choice process.

In a related paper, Ottaviani and Sorensen (2005) compare the performance of parimutuel and posted-offer betting mechanisms. Unlike our paper, they analyze the two setups separately and they leave the coexistence of these two mechanisms in one market to future research. They consider a race between two horses. There are three types of agents; bookmakers, a group of naive bettors and a group of sophisticated bettors. Naive bettors follow fixed betting rules. Bookmakers and sophisticated bettors share a common prior belief about the outcome of the race. Sophisticated bettors receive an extra private signal, and hence on average are better informed than the bookmakers. Bookmakers competitively set the odds (in equilibrium, they make zero profits) and bettors decide whether to bet a fixed amount on one of the horses or to abstain. They characterize the equilibrium prices (i.e. odds) and conclude that the expected return on longshots is decreasing in the number of sophisticated bettors.

Finally, Manski (2006) finds that equilibrium prices in prediction markets is a function of bettors' beliefs, risk preferences and their endowments. Wolfers and Zitzewitz (2005) and Gjerstad(2005) also reach the same conclusion. This result is supported by our model as well.

In terms of the second strand of literature our paper can be categorized into, Rust and Hall (2003) presents a model where consumers and producers choose who to trade with. They can trade with market makers, who publish their prices publicly, or they can choose to trade with middlemen, which involves some costly search. Rust and Hall show that highest-valuation consumers and lowest-search producers trade with the market maker whereas others search for better prices through dealing with the middlemen. Due to the different structure of Rust and Hall's model, how much trading volume middlemen keeps is dependent on the intertemporal discount rate, per unit transaction cost of the market maker and that of the most efficient middleman. They also show that coexistence of the market maker along with middlemen depends on the comparison of their per unit transaction cost parameter values.

Gehrig (1993) also investigates the coexistence of an intermediated market and a search market. Prices are posted for everyone to observe in the intermediated markets but trading with them has a transaction cost. On the other hand, trading in the search market requires being matched by a random matching technology as well as negotiating the price with the counterpart once matching is successful. Both markets exist as long as intermediated markets can charge a positive bid-ask spread and that there is enough heterogeneity among traders. Similar to other findings, traders who do not have much to gain from trading choose to go to the search market whereas others trade at the intermediated market.

In a recent article, Neeman and Vulkan (2010) presents a model of competition between a decentralized bargaining market and a centralized market. This study finds that these two markets do not coexist and trade takes place in the centralized market. Buyers and sellers in the model prefer to trade in the centralized market because price does not move against them as much as it does in the decentralized market.

Compared to the models mentioned above, the approach in our model is a different one. We do not have transaction costs or time costs in the model, but there is uncertainty in terms of knowing what price will be offered at the betting exchange. Our bettors have subjective beliefs about the outcome of the event they are betting on and they are risk-averse. In fact, we find that the risk aversion level of the bettors in our model strongly determines whether coexistence is achieved or not. Similar to some of the results from the literature, we also find that bettors with high subjective beliefs choose to place their bets with the bookie when both markets coexist.

3 Model

3.1 Setup

The model incorporates both of the betting institutions discussed above. The noble feature of this model is that bettors have a choice. There is a single two-outcome (win or lose) game played between teams (or players) K (Kings) and L (Lakers). The winning team is identified with $x = \{K, L\}$. There are two betting institutions: a bookmaker (or bookie) and a betting exchange. The bookie posts fixed odds for each outcome. The betting exchange, on the other hand, is a platform where bettors can submit buy or sell orders for the outcome $x = K$ only. A market-maker then collects all the orders and sets a market-clearing price p_K. We assume that price determination in the betting exchange is subject to some uncertainty, so bettors cannot perfectly foresee the trading price. This may be due to many reasons. There may be a number of naive bettors whose betting behavior cannot be predicted beforehand. It is not hard to find people like this in prediction markets, especially in sports betting. Many people bet on the team they support without any logical reason such as profitable odds. Alternatively, bettors may be uncertain about the aggregate distribution of beliefs so they cannot precisely predict the price that would arise in the betting exchange.

A continuum of sophisticated bettors choose to bet on their choice of the winner with a bookie, or choose to place buy or sell orders in the betting exchange. Each bettor is characterized with a subjective belief that K is the winning team, denoted with s, and a total initial wealth, denoted with w. The subjective beliefs of the sophisticated bettors are distributed over $[0, 1]$ according to a probability distribution function $F(s)$, and their initial wealths are distributed over $[\underline{w}, \overline{w}]$ according to a probability distribution function $H(w)$. Bettors compare the utility they would get from placing their bets with the bookie with the utility they expect to get from placing their bets in the betting exchange. A single bettor's bet does not influence the price in the betting exchange.

We assume that the bookie chooses his odds such that the resulting expected demands for the two outcomes are equal. We initially assume, in line with the previous literature, that the bookie behaves competitively so that he earns zero expected profits in equilibrium. We later allow the bookie to make positive profits by setting non-competitive prices. Let $b_K \in (0, 1)$ denote the price (or odds) the bookie chooses for an asset which pays back \$1 if the outcome is K, and 0 otherwise. By the assumption of zero profits, $1 - b_K$ is the price of an asset which pays back \$1 if the outcome is L, and 0 otherwise.

If a bettor purchases q units of the $x = K$ asset at a price b_K, then her net earning is $(1 - b_K)q$ in state $x = K$ and $-b_K q$ in state $x = L$, whereas the bookie's net earning on this transaction is $q(b_K - 1)$ in state $x = K$ and qb_K in state $x = L$.

The timing of our model is as follows. First, the bookie chooses his odds taking into account the optimal behavior of the sophisticated bettors, and the distribution of their beliefs and wealths. Sophisticated bettors observe the bookie's odds, and decide whether to place their bets with the bookie or place an order in the betting exchange. We assume that once a bettor decides to place an order in the betting exchange, it is too costly for her to go back to the bookie later. This is easily justified by the fact that it is typically very difficult (or sometimes impossible) in many online betting websites to liquidate a deposit that has already been made. Bettors who choose the betting exchange place their orders there and after all the orders are received (including the uncertain demand), a market-maker determines the market-clearing price.

3.2 Bettor Behavior

We first describe the optimal behavior of the sophisticated bettors. Assume that all bettors have the constant relative risk aversion (CRRA) utility function with $\theta > 0$ being the risk aversion parameter.

$$u(w) = \begin{cases} \frac{w^{1-\theta}}{1-\theta}, & \text{if } \theta \neq 1, \\ \log w, & \text{if } \theta = 1. \end{cases} \tag{1}$$

Suppose bettors can choose to buy or sell the $x = K$ asset at a price p. If this is the price offered by the bookie, then selling the $x = K$ asset at a price p means purchasing the $x = L$ asset at a price $1 - p$. Given her subjective belief s_i, bettor i chooses the optimal amount of $x = K$ asset to hold, q_i, that maximizes her expected utility. A positive value of q_i means purchasing the $x = K$ asset, while a negative value means selling the $x = K$ asset, or equivalently, purchasing the $x = L$ asset. Bettor i's problem is to choose a quantity to maximize

$$U(q_i, p, s_i, w_i) = \begin{cases} s_i \frac{(w_i + q_i(1-p))^{1-\theta}}{1-\theta} + (1 - s_i)\frac{(w_i - q_i p)^{1-\theta}}{1-\theta}, & \text{if } \theta \neq 1, \\ s_i \log(w_i + q_i(1-p)) + (1 - s_i)\log(w_i - q_i p), & \text{if } \theta = 1, \end{cases} \tag{2}$$

where $U(q_i, p, s_i, w_i)$ is the expected utility function.

The first order condition for this maximization is

$$q^*(p, s_i, w_i) = \frac{\left((1-p)^{1/\theta} s_i^{1/\theta} - p^{1/\theta}(1-s_i)^{1/\theta}\right) w_i}{(1-p)p^{1/\theta}(1-s_i)^{1/\theta} + p(1-p)^{1/\theta} s_i^{1/\theta}}. \tag{3}$$

Hence, bettor i's indirect (expected) utility function can be expressed as $U^*(p_K, s_i, w_i) = U(q^*(p, s_i, w_i), p, s_i, w_i)$. Equation (3) tells us that a bettor will bet on team K when his subjective belief of team K winning is higher than the price, $s_i > p$. He will be on team L (i.e., sell team K) if otherwise. He will not bet if $p_K = s_i$.

3.3 Price determination in the Betting Exchange

Suppose that, in equilibrium, sophisticated bettors with subjective beliefs $s \in S_E$ choose to place their bets in the betting exchange. The betting exchange also receives an ex-ante uncertain number of bets placed by other bettors (e.g. naive bettors). For simplicity, we assume that these bets do not depend on the price. Let ε denote the total number of buy orders for the $x = K$ asset placed by these bettors. Everyone shares the same prior beliefs for ε which is described by a symmetric probability distribution function $G(\varepsilon)$ over the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$. So, the expected value of ε is zero.

The market-maker collects all buy and sell orders for the $x = K$ asset, and then determines a price, $p_K(\varepsilon)$, at which the total buy quantity is equal to the total sell quantity:

$$\int_{\underline{w}}^{\overline{w}} \int_{s \in S_E} q^* \left(p_K(\varepsilon), s, w\right) dF(s) dH(w) + \varepsilon = 0. \tag{4}$$

Equation (4) uniquely determines a market-clearing price, $p_K(\varepsilon)$, which depends only on the realization of ε. We can alternatively interpret ε as the fact that no bettor knows about the actual total demand in the exchange.[6] Even if a bettor takes into account the total demand arising from the beliefs of others, there might still be demand motivated by things other than subjective beliefs in the market, such as bets placed by fanatics.

3.4 Bookie Behavior

Suppose, in equilibrium, $S_E \neq [0, 1]$, so that a positive measure of sophisticated bettors, i.e., those with subjective beliefs $s \notin S_E$, choose to bet with the bookie. We assume that the bookie does not face any uncertainty regarding the number of bets received. Even if we allow a random fraction of the uncertain bets to be placed with the bookie, this would not change our results since ε is symmetric around zero. The bookie's problem is to set a price b_K for the $x = K$ asset such that the total quantity bet on team K equals the total quantity bet on team L.

$$\int_{\underline{w}}^{\overline{w}} \int_{s \notin S_E} q^* \left(b_K, s, w\right) dF(s) dH(w) = 0. \tag{5}$$

Equation (5) uniquely determines a market-clearing price b_K. At this price, the bookie makes a sure return of zero.

4 Equilibrium

Given two different institutions to bet with, we can reach three different scenarios. The scenario we are interested in is when a proportion of bettors place their bets with the bookie and the remaining proportion with the betting exchange (coexistence of these two markets). In addition to that, two alternative scenarios are that either every sophisticated bettor places their bets with the bookie, or all place their bets in the betting exchange. As we will argue later, the first of these alternative scenarios never arises whereas we describe the conditions under which the latter can happen.

An equilibrium with coexistence requires that bettors placing their bets in the betting exchange are better off relative to the alternative of placing their bets with the bookie. A bettor is better off in the betting exchange if

$$\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} U(p_K(\varepsilon), s_i, w_i) dG(\varepsilon) \geq U(b_K, s_i, w_i). \tag{6}$$

In an equilibrium with coexistence of the two betting institutions, we need only a subset of bettors, $S_E \subset [0,1]$, placing their bets in the betting exchange, as well as equation (6), to be satisfied for all $s_i \in S_E$ (and the reverse inequality holds for all $s_i \notin S_E$).

An analytical closed-form solution to the described equilibrium does not exist since there is no closed-form solution to the integral $\int_{s \in S_E} q^* (p_K(\varepsilon), s, w) \, dF(s)$. Establishing coexistence analytically also proves to be complicated. Instead, we motivate the coexistence of these two market institutions using a graphical explanation. We then numerically simulate the model to characterize the equilibrium and identify the conditions under which these two institutions coexist. We also allow the bookie to earn positive profits by setting non-competitive prices, and show that this is almost always possible.

We first note that the risk aversion parameter, θ, needs to be sufficiently high for the coexistence of the two market institutions. When θ is low, bettors enjoy price uncertainty even though they are risk averse to wealth uncertainty. This is easiest to see when $\theta = 1$.

When the $x = K$ asset is offered at a price p, a bettor with a subjective belief s_i and an initial wealth w_i chooses a quantity $q(p, s_i, w_i) = \frac{(s_i - p)w_i}{p(1-p)}$, and accordingly, her indirect expected utility can be expressed as

$$\begin{aligned} U(p, s_i, w_i) &= s_i \log\left(w_i + \frac{(s_i - p)w_i}{p(1-p)}(1-p)\right) + (1 - s_i)\log\left(w_i - \frac{(s_i - p)w_i}{p(1-p)}p\right), \\ &= s_i \log\left(\frac{s_i w_i}{p}\right) + (1 - s_i)\log\left(\frac{(1 - s_i)w_i}{(1-p)}\right). \end{aligned}$$

The second derivative of this expression with respect to price is $\frac{s_i}{p^2} + \frac{1-s_i}{(1-p)^2}$ which is positive for all subjective beliefs. So, bettors are risk-loving when it comes to price uncertainty. This means that all bettors prefer the betting exchange to the bookie when the bookie offers a price equal to the expected value of the price that may arise in the betting exchange, i.e., when $b_K = \int_{-\overline{\varepsilon}}^{\overline{\varepsilon}} p_K(\varepsilon) dG(\varepsilon)$.

This is graphically illustrated in Figure 1 for $\theta = 1$. We capture price uncertainty in the betting exchange by taking p_K uniformly distributed over $[0, 1]$. The bookie offers a price $b_K = E(p_K) = 0.5$. We take a single bettor with a belief s and an initial wealth $w = 1$. The solid curve indicates the bettor's indirect expected utility when she bets with the bookie, i.e., $U^*(0.5, s, 1)$. Similarly, the dashed curve indicates the same when she bets in the betting exchange, i.e., $\int_0^1 U^*(p, s, 1)dp$. Since the dashed curve is above the solid one for all subjective beliefs, all bettors prefer the betting exchange, so the bookie cannot steal any bettors away. If the bookie increases the price he offers, the solid curve shifts to the right. Even though the bettors with high beliefs now choose the bookie over the betting exchange, the bookie does not want to do this since he cannot equalize the expected demands for the two outcomes. As a result, we conclude that when θ is not sufficiently high, all sophisticated bettors place their bets in the betting exchange.

[FIGURE 1 HERE]

When $\theta > 2$, both the betting exchange and the bookie may coexist in the betting market. This result is generated directly from two observations. First, the bettors with the strongest beliefs (i.e., $s = 0$ and 1) invest all of their wealth in the team they support. Take the bettor with $s = 0$. From equation (3), her demand is $q = -\frac{w}{1-p}$, and accordingly, her expected utility is $U(p, s, w) = \frac{(w/(1-p))^{1-\theta}}{1-\theta}$. The second derivative of this expression with respect to p is $(2-\theta)w^{1-\theta}(1-p)^{\theta-3}$, which is negative at all prices when $\theta > 2$. A symmetric result holds for $s = 1$. So, when $\theta > 2$, the bettors with the strongest beliefs choose the bookie if the bookie offers a price that is equal to the expected value of the price that may arise in the betting exchange. The second observation is that the bettor with a belief that perfectly coincides with the bookie's price, $s = b_K$, always chooses to bet in the betting exchange since her expected surplus is positive there while it is zero with the bookie. The remaining bettors may or may not enjoy price uncertainty depending on the exact price range.

We illustrate these observations in Figure 2 for $\theta = 5$. We take a single bettor with a belief s and an initial wealth $w = 1$. Each curve describes the bettor's expected (indirect) utility when she bets her optimal amount at a price p. The thick solid curve corresponds to the case when the bettor has a belief $s = 0$, the thin solid curve corresponds to $s = 0.05$ and the dashed curve to $s = 0.5$. When $s = 0.5$, the bettor's expected utility is convex in price at all prices. Similarly, when $s = 0$, it is concave in price at all prices. When $s = 0.05$, on the other hand, the bettor's expected utility is convex when price is close to her belief and slightly concave when price is away. So, when the range of possible prices in the betting exchange is, say, $[0.4, 0.6]$, bettors whose beliefs are neither so strong, nor moderate ($s = 0.05$) do not like price uncertainty. However, when the range expands, the convex part in these bettors' utility function starts dominating, and therefore more bettors place their bets in the betting exchange.

[FIGURE 2 HERE]

Bettors with strong subjective beliefs do not benefit as much from price uncertainty as those with more moderate beliefs. This is because their position as a buyer or seller does not depend very much on the realization of the price. They also invest a high fraction of their wealth in the team they support (the whole wealth when $s = 0$ and 1), and the amount of the $x = K$ (or L) asset they hold responds too strongly to price movements. The bettors with moderate beliefs, on the other hand, are more likely to have their positions (as a buyer or seller) changed depending the realization of the price. On average, this is better than buying or selling a low quantity at a fixed price.

We reanalyze the earlier example when bettors are sufficiently risk-averse, $\theta = 5$. As before, we take p_K uniformly distributed over $[0, 1]$ and $b_K = E(p_K) = 0.5$. As seen in Figure 3, the solid curve is now above the dashed one at the tails, so the bookie can successfully steal the bettors with strong beliefs away from the betting exchange.

[FIGURE 3 HERE]

Note that Figure 3 does not illustrate an equilibrium although the bookie is able to equalize the demands for the two outcomes. This is because the price in the betting exchange and the price the bookie charges are endogenous. When the bookie steals away some bettors, the betting exchange is no longer in an equilibrium with prices uniformly distributed over $[0, 1]$ because a higher fraction of the price is determined by the uncertain demand now.

Based on the discussion so far, we posit that in any coexistence equilibrium, only the bettors with sufficiently strong beliefs choose to bet with the bookie. We need to find a stable point at which the equilibrium conditions (i.e., equations (4), (5) and (6)) are satisfied. We first numerically simulate the model when $\bar{\varepsilon} = 0.2$ and ε takes only three values, -0.2, 0 and 0.2, with equal probabilities, and the sophisticated bettors' beliefs have a truncated normal distribution over $[0, 1]$ with a mean of $\mu = 0.5$ and a standard deviation of $\sigma = 0.2$. We take $E(w) = 1$ and $\theta = 5$. Since both s and ε are symmetric around 0.5, the equilibrium displays symmetry as well. The bookie sets a price $b_K = 0.5$ to equalize the expected demands for the two outcomes. We need to find one threshold belief, $\bar{s}$, such that the bettors with subjective beliefs $\bar{s}$ and $1 - \bar{s}$ are indifferent between betting with the bookie and the betting exchange, i.e., $E_{\varepsilon}[U(p_K(\varepsilon), \bar{s}, w)] = U(b_K, \bar{s}, w)$.

We find that, for these parameter values, $\bar{s} \approx 0.0684$, only a small fraction (approximately 1.85%) of the sophisticated bettors choose to bet with the bookie. This, however, is due to the assumption of normally distributed beliefs. The bookie could have a substantial market share if we had a quasiconvex belief distribution symmetric around 0.5. Since ε takes only three possible values, so does the equilibrium price in the betting exchange. We find that p_K equals 0.279, 0.5 or 0.721 with equal probabilities.

Our numerical simulations indicate that, when bettors' beliefs are symmetric around 0.5, a coexistence equilibrium exists for any (symmetric) distribution of ε as long as $\theta > 2$. This is when the bettors with the strongest beliefs (i.e., $s = 0$ and 1) are strictly risk-averse to price uncertainty. So, even an infinitesimal amount of noise in the betting exchange induces coexistence when the bettors are sufficiently risk-averse. On the other hand, when $\theta \leq 2$, all bettors place their bets in the betting exchange.

Result 1 *When $\theta > 2$, the two market institutions coexist for all distributions of bettor beliefs that are symmetric around* 0.5*. When $\theta \leq 2$, the betting exchange offers higher expected payoff to all sophisticated bettors.*

Next, we analyze how the magnitude of price uncertainty in the betting exchange (captured by $p_K(\overline{\varepsilon})$) and the fraction of the sophisticated bettors that choose to bet with the bookie (captured by $\overline{s}$) change with respect to four exogenous parameters, $\overline{\varepsilon}$, θ, $E(w)$ and σ, while maintaining the symmetry of beliefs and ε. Our findings are summarized in Result 2.

Result 2 *We find that $p_K(\overline{\varepsilon})$ is increasing in $\overline{\varepsilon}$, θ and σ, and decreasing in $E(w)$, while $\overline{s}$ is decreasing in $\overline{\varepsilon}$ and σ, increasing in w, and increasing in θ up to approximately $\theta = 10$ and decreasing afterwards.*

We also provide these findings in Table 1.

TABLE 1. Comparative statics analysis:

Impact of changes in $\overline{\varepsilon}$, θ, $E(w)$ and σ on $p_K(\overline{\varepsilon})$ and $\overline{s}$.

	$\overline{\varepsilon}$	θ	$E(w)$	σ
$p_K(\overline{\varepsilon})$	$+$	$+$	$-$	$+$
$\overline{s}$	$-$	$+,-$	$+$	$-$

Expectedly, as $\overline{\varepsilon}$ increases, the range of possible prices in the betting exchange (i.e., $p_K(\overline{\varepsilon}) - p_K(-\overline{\varepsilon})$) expands. This is because a higher realization of ε means an increase in the demand for the $x = K$ asset, and a higher price is required to clear the market. As price becomes more uncertain in the betting exchange, fewer bettors bet with the bookie. The intuition for this result comes from Figure 2. As explained earlier, a bettor with a belief $s = 0.05$ may dislike price uncertainty when its range is narrow. But, as price becomes more uncertain, her risk-attitude towards price uncertainty changes. When the price becomes sufficiently uncertain, she starts enjoying it.

Interestingly, the fraction of bettors who bet with the bookie is non-monotonic in the risk-aversion parameter. It is increasing up to $\theta \approx 10$, and decreasing afterwards. The reasoning for this finding is as follows. When θ is low, the range of possible prices in the betting exchange is too narrow, because most of the bettors place their bets in the betting exchange (all of them when $\theta = 2$) and they invest a high proportion of their wealths in their bets. As θ increases, initially both the number of bettors in the betting exchange and the quantities they are willing to trade go down. So, $\bar{s}$ initially goes up. Since ε becomes more important in price determination in this case, price becomes more uncertain. But as price becomes more uncertain, some bettors who disliked price uncertainty start enjoying it and switch back to the betting exchange. This effect dominates when θ is sufficiently high even though bettors trade lesser quantities.

As bettors become more wealthy on average, they trade higher quantities at all prices. Thus, sophisticated bettors' aggregate demand becomes more important in the determination of price in the betting exchange, and therefore, the range of possible prices shrinks. As price becomes less uncertain, more bettors start disliking price uncertainty and switch to the bookie. So, more bettors bet with the bookie. Note that the wealth effect disappears if ε also changes by the same factor.

Finally, as the belief distribution of sophisticated bettors become more uniform, the mass of beliefs between $\bar{s}$ and $1-\bar{s}$ goes down. So, for a given $\bar{s}$, there are now fewer bettors in the betting exchange. In this case, ε becomes more important in the determination of price in the betting exchange, and therefore, the range of possible prices expands. When price becomes more uncertain, some bettors switch away from the bookie and start placing their bets in the betting exchange. So, $\bar{s}$ goes down.

Next, we allow the bookie to make positive profits by increasing the price by 0.01 for each event, i.e, by setting $b_K = b_L = 0.51$. Obviously, fewer bettors place their bets with the bookie, but the bookie still gets a positive demand for both outcomes. In this case, we find that $\bar{s} \approx 0.04$. The threshold belief $\bar{s}$ decreases as the price mark-up increases and eventually reaches zero when $b_K = b_L \approx 0.5642$. Thus, a bookie with some market power can choose a non-competitive price between 0.5 and 0.5642, and make positive profits. When this is the case, the betting exchange on average offers better prices than the bookie for both outcomes. Even in this case, a coexistence of the two market institutions is possible. Clearly, the bookie enjoys higher profits when more bettors bet with him, i.e., when $\bar{s}$ is higher.

Result 3 *In a coexistence equilibrium, the bookie may make positive profits by setting both prices above* 0.5*. Furthermore, the profit margin is decreasing in the magnitude of uncertain demand (i.e., in* $\bar{\varepsilon}$*) and in the average wealth, while it is decreasing in the standard deviation of the belief distribution.*

This observation has important implications. If we start from a benchmark situation where the bookie serves the whole market of sophisticated bettors and possibly charges non-competitive prices, the entry of a betting exchange would steal most of the bettors away from the bookie. In this case, the bookie will have to decrease the price mark-up that he used to charge. So, even when we allow the bookie to make positive profits, having betting exchange is unarguably good for all bettors.

Result 4 *Compared to a benchmark situation where there is only a bookie, bettors who choose to bet in the betting exchange are strictly better off while those who choose the bookie are not worse off.*

When beliefs are symmetric around 0.5, we can make strong predictions. It is more difficult to make strong predictions when beliefs are asymmetric because it is computationally timely to converge to an equilibrium. We have simulated the model when s has a truncated normal distribution with a mean $\mu = 0.25$ and a standard deviation $\sigma = 0.2$, while all the other parameters are the same as in the symmetric example. We find that there is an equilibrium with coexistence in which bettors with beliefs $s \in S_E = [0.0042, 0.756]$ place their bets in the betting exchange while the remaining (approximately 0.93% of the sophisticated bettors) bet with the bookie. The bookie charges $b_K = 0.273$ and the possible prices in the betting exchange are 0.122, 0.258 and 0.482 with a mean price 0.287. So, on average, the bookie offers a lower price for the outcome $x = K$ and a higher price for $x = L$.

We then introduce positive profits by assuming that the prices bookie charges satisfy $b_K + b_L = 1.03$, so when the bookie equalizes the demands for the two outcomes, there is a sure profit of 0.015 for each unit sold. There is still an equilibrium with coexistence. Bettors with beliefs $s \in S_E = [0.0003, 0.908]$ place their bets in the betting exchange, so the bookie still gets a positive demand (approximately 0.07%). The bookie's prices are $b_K = 0.291$ and $b_L = 0.739$ while the possible prices in the betting exchange are 0.123, 0.258 and 0.479 with approximately the same mean as before. So, in this case, the bookie offers higher prices for both outcomes.

We conclude that, when bettor beliefs are not symmetric around 0.5, a coexistence equilibrium generally exists even though the betting exchange on average may offer better (i.e., lower) prices for both outcomes. Our simulations indicate that the comparative statics results we found earlier for symmetric beliefs still hold.

5 Conclusion

In this paper, we create a theoretical model that examines the coexistence of two competing betting mechanisms in a single market; one mechanism follows the posted-offer rule and the other one incorporates a double-auction format. We motivate the study of these markets with a sports betting example. In the model, bettors are free to choose which mechanism they want to place their bets with. Since bettors are given the freedom to choose, we find that bettors' level of risk aversion is instrumental in whether these two markets coexist or not. When bettors' CRRA risk aversion parameter is high enough (larger than 2), both mechanisms coexist and bettors with sufficiently strong subjective beliefs choose to place their bets with the posted-offer mechanism (bookie). For low values of this parameter, we find that all bettors choose to place their bets with the betting exchange and the bookie goes out of business. These results hold even when we allow the bookie to make positive profits instead of following a zero profit pricing rule.

We also show that compared to a benchmark situation where only a bookie is available to take bets, allowing bettors to choose which mechanism to bet with makes those who choose the betting exchange strictly better off while those who choose the bookie not worse off. This means allowing both mechanisms to coexist is beneficial to bettors in general. A large proportion of bettors choose the betting exchange over the bookie because they get higher expected utility from that choice. As a result, in the case of the U.S., we hypothesize that removing the legal obstacles to online sports betting would allow betting exchanges to enter this market, making bettors better off. We propose that countries such as the U.S., France, Turkey, and the Netherlands should consider changing their laws such that bettors in those countries can have easy access to betting exchanges as well.[7]

Notes

[1]While there are tedious ways of transmitting your funds to these overseas sportsbook companies, these methods are not easy enough to generate a market for these companies in the U.S..

[2]Even though our motivation uses a law from the U.S., other countries such as Turkey, France or the Netherlands have different laws resulting in the same outcome as well. On the other hand, countries such as England and the Czech Republic allow for coexistence of these two mechanisms in the same market.

[3]See, for example, Ketcham et al. (1984) and Hong and Plott (1982).

[4]There are many betting exchanges available for bettors, especially in sports betting. The one exchange authors had in mind while writing the paper was Betfair. Betfair also has the biggest market share in England.

[5]The 3% commission is on winnings only. Therefore, on average the commission is only 1.5%. Moreover, the commission goes down as your cumulative bets are growing.

[6]This idea is similar to the noise traders used in the models of financial markets.

[7]Contrary to our proposal, France has just recently passed a law, that came into effect on May 13th, 2010, preventing bettors from France, or any of its territories, to access betting exchanges.

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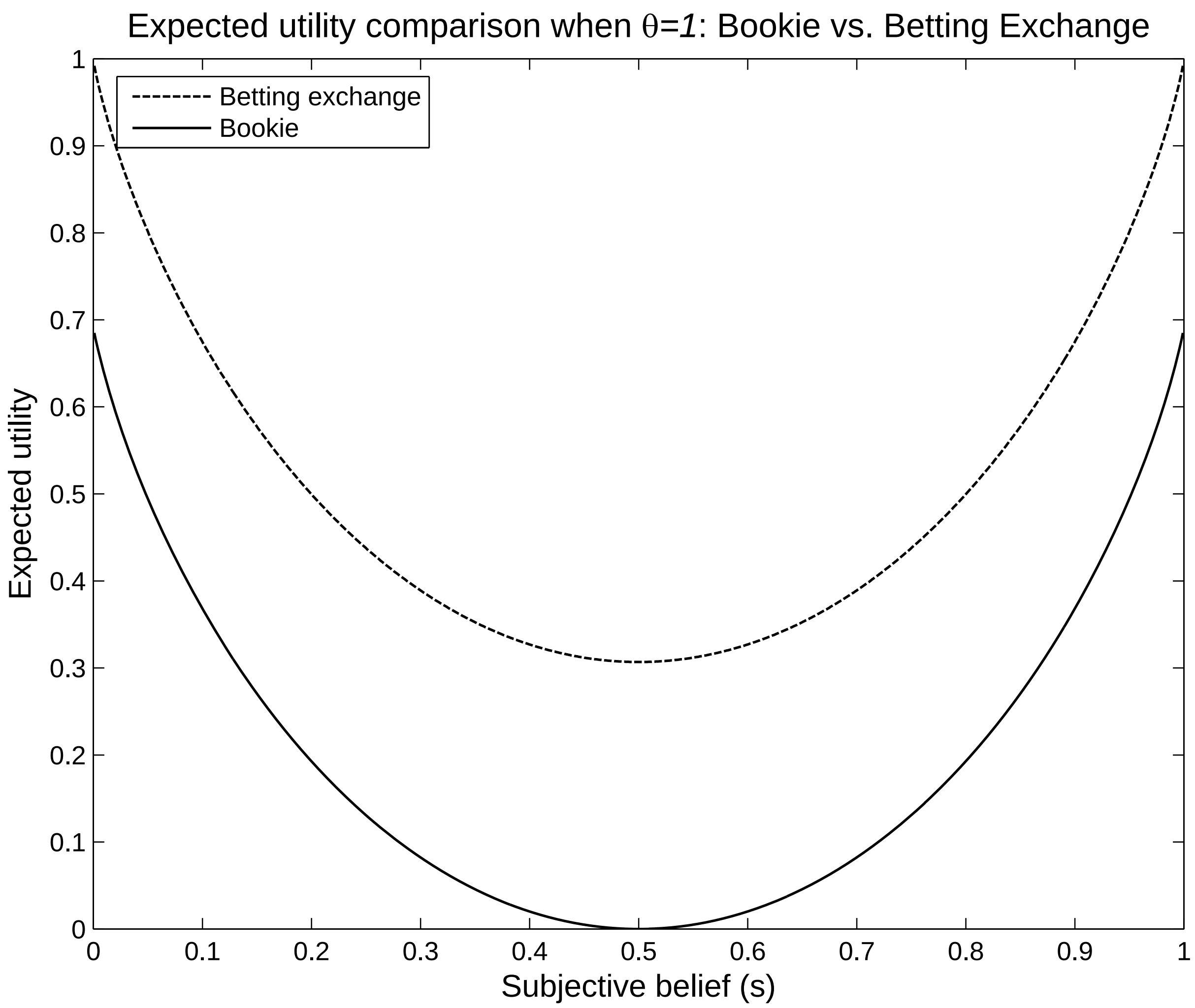

FIGURE 1.

The expected indirect utility of a bettor with a belief s and an initial wealth w=1 when the bookie offers a price 0.5 and the betting exchange has a price uniformly distributed over [0,1].

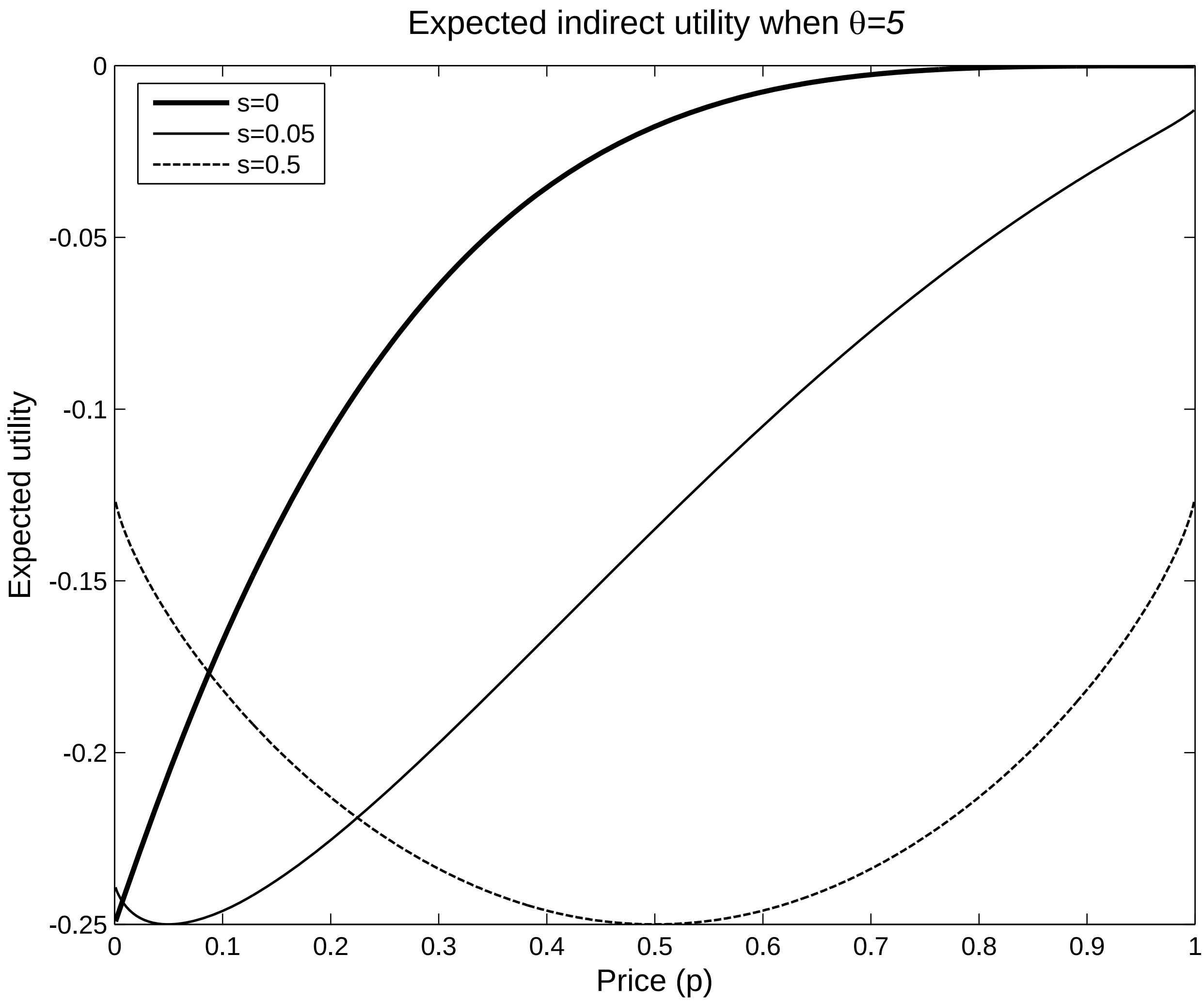

FIGURE 2.

Expected indirect utility of a bettor with a signal s and an initial wealth w=1 when she bets at a price p.

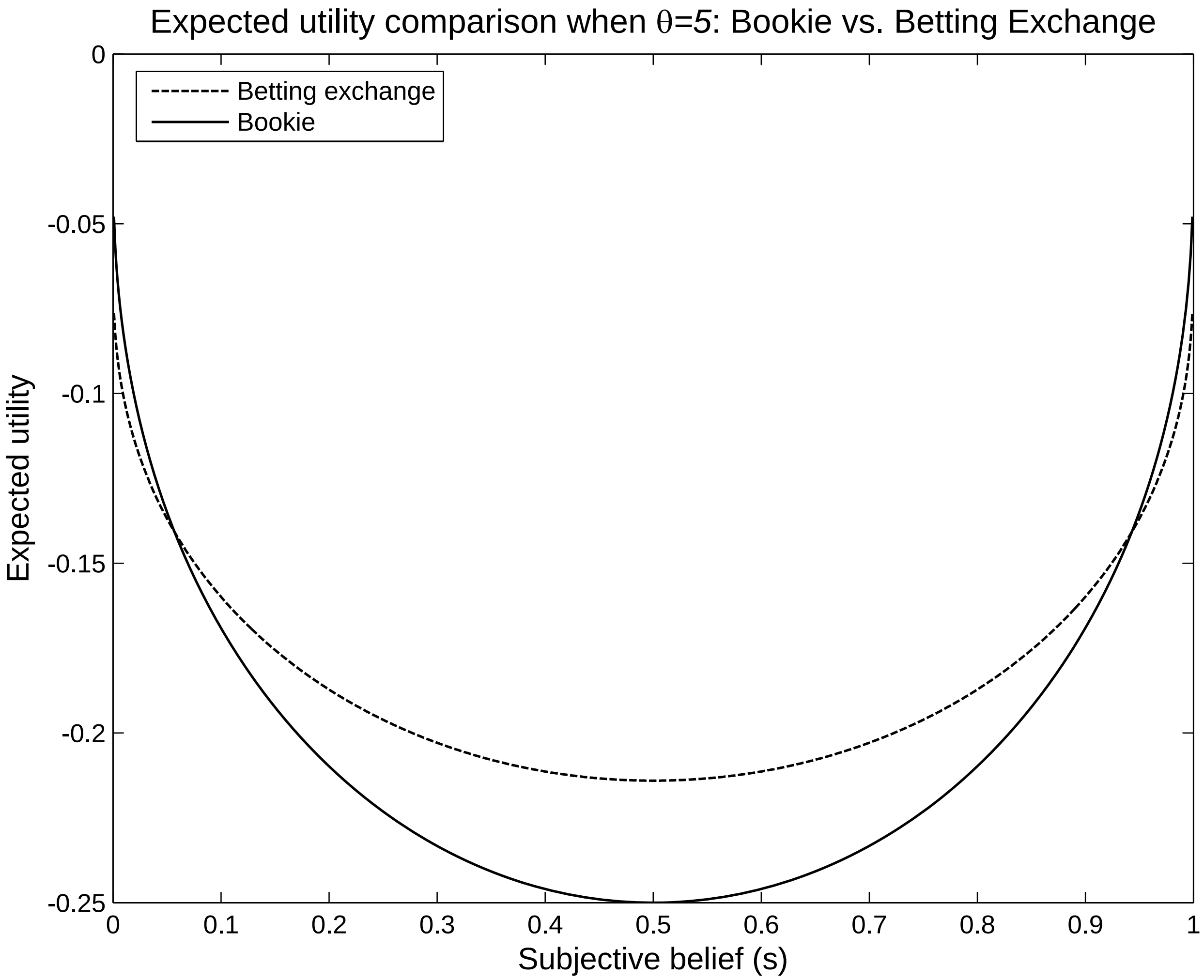

FIGURE 3.

The expected indirect utility of a bettor with a belief s and an initial wealth w=1 when the bookie offers a price 0.5 and the betting exchange has a price uniformly distributed over [0,1].