The Incomprehensible Effectiveness of Mathematics in Physical Sciences

A SIAM chapter colloquium presented at the Department of Mathematics & Statistics, CSUS

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Introductory Remarks

In a celebrated paper titled *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* written in 1960, Eugene Wigner, a distinguished twentieth century physicist wrote: "**. . . the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it**." Earlier, Albert Einstein had said: "**How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?**"

In this talk I will first briefly discuss the historical background to the above statements. Drawing on examples related to my own research, I will then attempt to illustrate the breadth of the application of mathematics in physics.

How Effective is Mathematics?

- Early mathematics: Numbers, Geometry, Algebra (Euclid, Khwarizmi)

- Issac Newton (1643-1727) & **Philosophi Naturalis Principia Mathematica** (1687), "most influential science book"

The foundations of classical physics, which would reach its pinnacle by the end of the 19th century, as well as differential and integral calculus, were essentially delivered in finished form in Newton's *Principia*.

- The Age of Reason and Pierre-Simon Laplace (1749-1827): *We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.*

- Was Laplace justified in making the above statement?

By the end of the 19th century, the rate at which the orbit of Mercury precesses was known accurately:

$$Observed\ Precession\ Rate = 5601\, arcsec/century = 1.556\, deg/century. \quad (1)$$

Newtonian mechanics (laws of motion plus gravity; coupled differential equations and perturbation theory) predicted

$$Newtonian\ Prediction = 5558\, arcsec/century = 1.544\, deg/century, \quad (2)$$

and the difference is

$$\begin{aligned} Difference &= 43\, arcsec/century = 0.012\, deg/century, \\ Relative\ Difference &= 8.0 \times 10^{-8}\, deg(precession)/deg(revolution). \quad (3) \end{aligned}$$

- Note the closeness of the above agreement, less than one tenth of one *ppm*.

- But there's more!
- Einstein's theory of gravitation (1915), general relativity, is based on the idea that space-time geometry is not Euclidean (flat) but Riemannian (curved).
- Paths of shortest distance (geodesics) near a massive body appear curved, e.g., they are very nearly elliptical around a star or a planet.
- Gravitation is thus geometrized, and the gravitational field is replaced by the metric tensor of space-time.
- Einstein showed that his theory of gravitation predicts an increase of precession rate for Mercury of 43 arc-sec/century!
- He also predicted that light from distant stars would be bent by the sun on the way to the Earth. This quantitative prediction was dramatically verified in 1919.
- The agreement on precession rates was now at the *ppb* scale!

Pure or Applied?

• Bernhard Riemann's work on non-Euclidian geometry would be considered "pure" mathematics for his day, though clearly not after 1915. Today, it is the nuts and bolts of our cosmology!

• Riemann, in his study of the distribution of prime numbers (1859), advanced the hypothesis that the non-trivial zeroes of his ζ-function would all lie on the line $\frac{1}{2} + iy$, where y is a real number.

• Can prime numbers be possibly relevant to the real world?

• Early in the 20th century, the laws of motion for microscopic objects (e.g., atoms and nuclei) were discovered and found to obey a non-commutative algebra.

• Thus position and momentum are represented by self-adjoint operators, respectively $\hat{x}$ and $\hat{p}$, on a Hilbert space (a complete metric space with a complex inner product) whose points correspond to possible states of the physical system. These operators obey the Heisenberg commutation condition, $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hat{1}$ (in units such that $\hbar = 1$).

- Around the turn of the century, Hilbert and Pólya conjectured that there exists a self-adjoint operator $\hat{H}$ whose spectrum coincides with the real part of the complex zeros of ζ, and, following the advent of quantum mechanics, Pólya speculated that $\hat{H}$ may be the energy operator of a physical system.

- At the end of the 20th century, Berry speculated that some self-adjoint extension of the operator $\frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2}$ might be the sought-after $\hat{H}$.

- This is where things stand at the moment, and the conjecture remains unresolved. However, the story of this “prime obsession” does not end here!

Primes in our lives

• Today, most commercial (as well as diplomatic/intelligence community) messages are secured using a public-key encryption system called RSA (after Rivest, Shamir, and Adleman).

• At the core of the RSA scheme is the computational difficulty of factoring a large number (e.g., 1024 bits long) into its prime factors (believed to belong to the NP computational complexity class).

• In 1994, Peter Shor published an algorithm which, used with a *quantum computer*, reduces the classical factorization problem (NP class) to a polynomial-time (P class) computation. The catch is the quantum computer!

• Amusingly, quantum theory, having potentially undermined the most secure public encryption system available, provides its own remedy, *quantum cryptography*, believed to be “absolutely” unbreakable!

• The basis of quantum cryptography is the non-commutative nature of dynamical observables in quantum theory.

Chaos, Randomness, and algorithmic Complexity

- At the end of the 19th century, Henri Poincaré realized that Laplace's promise could not be met in as simple a case as the gravitational three-body problem.

- By the seventies and eighties it was widely recognized that a typical deterministic, nonlinear dynamical system (of three or more dynamical variables) would be likely to exhibit the property of sensitivity to initial conditions, thus rendering long-time predictability impossible.

- This ubiquitous phenomenon is now known as "chaos," and causes the long-time behavior of the system to be effectively random.

- Here's a simple model (known as the *logistic map*):

$$x_{n+1} = \mu x_n (x_n - 1), \tag{4}$$

where $\mu \in (0, 4]$ is a parameter, n is an integer, and $0 < x_n < 1$ for any n.

• The output of the logistic map for large n is a random sequence.

• What is a random sequence? Is the sequence of numbers in the decimal expression of the venerable number π random?

• Algorithmic complexity of an object such a sequence S of N binaries is defined by reference to the length of the shortest description of the sequence in some standard language. Let this shortest description be encoded as a binary sequence of length $K_N(S)$. Then, the algorithmic complexity of a sequence S is defines as

$$k(S) := \lim_{N\to\infty} K_N(S)/N. \tag{5}$$

• The property of sensitivity to initial conditions for chaotic systems causes an exponential growth of information production as the system evolves. The rate of that growth, known as the Kolmogorov-Sinai entropy, turns out to be the same as the algorithmic complexity of a the output sequence of the system, such as the $\{x_n\}$ of the logistic map.

Concluding Remarks

• Einstein once said: "**The most incomprehensible thing about the world is that it is comprehensible**". We may rephrase the latter part of this statement by saying that **The laws of Nature are susceptible to a high level of algorithmic compression**. This would seem to be a necessary condition for the incomprehensible effectiveness of mathematics in describing the physical world.

• Some contend that we invent the mathematics that suit our needs.

• But where do we get the capacity, the facility, to do mathematics?

• Here's an intriguing thought: A human young of a few months appears to have a sense of universal grammar, an innate, context-free, facility for linguistic structure, and understand linguistic recursion (Chomsky, 1955).